# Bol Loops in Nilpotent Alternative Loop Ring 

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## Research Article


#### Abstract

Let $F L$ be the alternative loop ring (associative or otherwise) of a loop $L$ of order $2^{n}$ over a ring of characteristic 2. Then, with respect to any radical property for which nilpotent rings are radical and rings with 1 are not, the radical of $F L$ is its augmentation ideal $\Delta(L)$ and this is nilpotent of dimension $2^{n}-1$. 2010 Mathematics Subject Classification: Primary 17D05; Secondary, 20N05, 16N40, 16S34. Keywords: Alternative ring, Bol loop, Nucleus, Center, Loop Commutator, Loop associator, Loop ring.


## Introduction

A loop is a binary system $(L, \cdot)$ with an identity element 1 in which given any two of three elements $a \cdot b=c$ in $L$, third is uniquely determined by the equation $a \cdot b=c$. This paper is concerned with Bol loops; that is, loops in which any one of the following two identities valid.
$x(y z \cdot y)=(x y \cdot z) y$ right Bol identity
$y(x y \cdot z)=(y \cdot x y) z$ left Bol identity
The loop ring of $l$ with coefficients in $R$, denoted $R L$, is
the free $R$-module with basis 1 and multiplication given by extending the multiplication in $L$ via the distributive laws.
$\sum_{l \in L} \alpha_{l} l+\sum_{l \in L} \beta_{l} l=\sum_{l \in L}\left(\alpha_{l}+\beta_{l}\right) l$
$\left(\sum_{l \in L} \alpha_{l} l\right)\left(\sum_{l \in L} \beta_{l} l\right)=\sum_{l \in L}\left(\sum_{n k<l} \alpha_{n} \beta_{l}\right) l$
In this paper, we investigate the case where the ring has characteristic 2 and extend to alternative loop rings a well known result of Jennings [5] for group rings by proving that the augmentation of order $2^{n}$ in characteristic 2 is a nilpotent ideal (of dimension $2^{n}-1$ ). This, of course, means that virtually all the familiar radicals of alternative rings coincide with the augmentation ideal. Goodaire and Parmenter [4] have studied the property of "Semisimplicity of alternative rings" and proved that nucleus and center in loop. Also they should that an alternative loop ring $R L$ is semi prime if and only if the group ring of the center of $L$ is semi prime by studying the properties of nil ideals. H. O. Plugfelder [6] introduced by "Quasigroups and loops". For example, the smallest Bol loop $-B\left(S_{3}, 2\right)-R A 2$, but not nilpotent. It does, however, contain an abelian group. This proves to be typical of $R A 2$ loops, as we show in section 3, and is the key to what we want to establish about RA2 loop algebras in this paper.

## Preliminaries

If $x, y, z$ are elements of an alternative ring, we denote the (ring) commutator of $x$ and $y$ by $[x, y]$ and the (ring) associator of $x, y$ and $c$ by $[x, y, z]$.

Thus
$[x, y]=x y-y x$ and $[x, y, z]=(x y) z-x(y z)$.
Each of these functions is skew-symmetric. The kleinfeld function is defined by
$f(x, y, z, w)=[x y, z, w]-y[x, z, w]-[y, z, w] x$.
It too is skew-symmetric. (our general reference for the theory of alternative rings is [7]). An alternative ring $R$ has a nucleus
$N(R)=\{n \in R /[n, R, R]=[R, n, R]=[R, R, n]=0\}$
$=\{n \in R /[n, R, R]=0\}$
(by skew-symmetry of the associator) and a center $Z(R)=\{c \in N(R) / a c=c a$ for all $a \in R\}$.
If $x, y, z$ are elements of a loop, we denote the (loop) commutator of $x$ and $y$ by ( $x, y$ ) and the (loop) associator of $x, y$ and $z$ by $(x, y, z)$. Thus
$x y=(y x)(y, x)$ and $x y \cdot z=(x \cdot y z)(x, y, z)$.
(It is often convenient to use dots instead of, or in addition to, parentheses to denote the order of multiplication in a nonassociative product, with the convention that juxtaposition takes precedence over a dot). A loop $L$ has a nucleus $N(L)$ and a center $Z(L)$ defined in a way completely analogous to the manner in which they are defined in an alternative ring. The traditional reference for the theory of loops has been Bruck's classic text [1], but there is now available a more modern book by Pflugfelder [6] which contains the basis facts about loops. The most fundamental properties of $R A 2$ loops are contained in the following restatement of theorem 2.9 of [2]).
Theorem 2.1: An RA2 loop is a Bol loop in which, given a triple $g, h, k$ of elements which do not associate, precisely one of the following occurs:
i) $g, h$ and $k$ commute pairwise and if $x, y, z$ are $g, h, k$ in some order,
$x y \cdot z=g h \cdot k$ and $x \cdot y z=g \cdot h k$;
ii) exactly one of $g, h, k$ commutes with the other two and, if this element is $g$,
$h g \cdot k=g h \cdot k=g \cdot k h=k \cdot g h=k \cdot h g=h k \cdot g$
and $h \cdot g k=g \cdot h k=g k \cdot h=k g \cdot h=k h \cdot g=h \cdot k g$;
iii) exactly one of $g, h, k$ commutes with neither of the order two and, if this element is $g$,
$g k \cdot h=k \cdot g h=k h \cdot g=h k \cdot g=g h \cdot k=h \cdot g k$
and $g \cdot k h=k g \cdot h=k \cdot h g=h \cdot k g=g \cdot h k=h g \cdot k$;
iv) no pair of elements of the triple $g, h, k$ commute and, if $x, y, z$ are $g, h, k$ in some order,
$x y \cdot z=x \cdot z y=y \cdot x z$.
We refer to a $g, h, k$ of elements which do not associate as a triple of type (i), (ii) $g$, (iii) $g$ or (iv),
According as these elements satisfy (i), (ii), (iii) or (iv), respectively, of the theorem.
If this theorem appears somewhat complicated, it should be noted that we can often manage with a more easily remembered consequence of it.
Corollary 2.1: If $g, h$ and $k$ are three elements of an RA2 loop which do not associate, then either
$g h \cdot k=h g \cdot k$ and $g \cdot h k=h \cdot g k$
$g h \cdot k=h \cdot g k$ and $g \cdot h k=h g \cdot k$
according in as $g$ and $h$ do not commute, respectively.

## 3. RA2 loops

In this section, we establish a number of properties of RA2 loops, virtually all of which are generalizations of known results for RA2 loops.
Theorem 3.1: Let $L$ be an RA2 loop and let $g$ and $h$ be elements of $L$ such that $(g, h, L)=1$ (that is, $(g, h, k)=1$ for all $k \in L$ ). Then
i) $(g, h)^{2}=1$,
ii) $g^{2} h=h g^{2}\left(\right.$ and $\left.g h^{2}=h^{2} g\right)$, and
iii) The commentator $(g, h)$ commutes with $g$ (and with h).

Proof: The loop $L$ is embedded in an alternative ring $R L$ of characteristic 2 , we see that $(g, h g)=1 \Rightarrow[g, h, R L]=$ 0 . Then, following E.G. Goodaire [3, theorem 3 (iii)]), we obtain $n=g h+h g \in N(R L)$ and then, nothing that we also have $(g, h g, L)=1$, that $[g, h g]=n g$ is also in $N$ $(R L)$. (Here, we begin to use freely the fact that we are working in characteristic 2). It follows that $[n g, x, y]=0$ for all $x, y \in R L$ and so from (3),
$0=g[n, x, y]+[g, x, y] n+f(n, g, x, y)=[g, x, y] n$
Since $n \in N(L)$ and $f(n, g, x, y)=f(g, x, n, y)$ is the sum of three terms each of which involves an associator containing $n$. Assume for the moment that $g \in N(L)$. Then $g^{2} \in Z(L)$ and $(g, h)^{2}=1$ by [2, corollaries 3.5 and 3.6], so (a) and (b) of the theorem hold. As for (c), this holds if $N$ $(L)$ is not commutative by [2, theorem 3.2], so we now consider the implications of a commutative nucleus. In this case, and if, furthermore, $N(L)=Z(L)$, then statement (c) holds since we would then have $g \in Z(L)$. Finally, if $N(L) \neq Z(L)$ (but still assuming the nucleus is commutative), then $L$ is an Bol loop [8] in which squares, and hence commentators, are in the nucleus (because ( $a$, $b)=a^{-1} b^{-1} a b=\left(a^{-2}\left(a b^{-1}\right)^{2} b^{2}\right)$. So $g$ and $(g, h)$, being in the nucleus, must commute. The theorem is therefore true if $g \in N(L)$. It remains to consider the case that $g \notin N(L)$. In this case, choose $a$ and $b$ in $L$ with $(g, a, b) \neq 1$. Write $g a$.
$b=(g \cdot a b) k$ for $k=(g, a, b) \in L$ and note that $[g, a, b]=$ $g a \cdot b+g \cdot a b=(g \cdot a b)(k+1)$. Then, from (6), $[g, a, b] n$ $=0$, so $(g \cdot a b)(k+1) n=0$ and because $g \cdot a b$ is invertible, $(1+k) n=0$. Recalling that $n=g h+h g$, we obtain $g h+$ $h g+k \cdot g h+k \cdot h g=0$. Now $g h \neq k \cdot g h$ since $k \neq 1$, so by linear independence of loop elements in the loop ring, either $g h=h g$, in which case the theorem is true trivially, or $g h=k \cdot h g$ and $h g=k \cdot g h$. In this last case, we have $g h$ $=k(k \cdot g h)$, implying that $k^{2}=1$. Moreover, $k=\left(g^{-1}, h^{-1}\right)$ and so
$K=(g, a, b)=\left(g^{-1}, h^{-1}\right)$.
Now, since $(g, h, L)=1$, we have also $(g, g h, L)=1$. Repeating the foregoing argument for $g$ and $g h$ and nothing that the theorem holds if $g$ and $g h$ commute, we may assume that $(g, a, b)=\left(g^{-1},\left(g h^{-1}\right)\right)$, so that $\left(g^{-1}, h^{-1}\right)=$ $\left(g^{-1},(g h)^{-1}\right)$. This immediately gives that $g$ and $\left(g^{-1}, h^{-1}\right)$ commute. It is not hard to show that this forces $g$ and ( $g$, $h$ ) to commute as well. Also, since $k$ has order 2, so does $(g, h)$. Hence $\left(g^{2}, h\right)=g^{-2} h^{-1} g^{2} h=g^{-1}\left(g^{-1} h^{-1} g\right) g h$ $=g^{-1}\left(g^{-1} h^{-1} g\right) h g(g, h)$
$=g^{-1}\left(g^{-1} h^{-1} g h\right) g(g, h)$
$=g^{-1}(g, h) g(g, h)=1$
and the theorem is complete. $\square$
Lemma 3.1: Let $g$ and $h$ be elements of an RA2 loop $L$. Then $g h=h g \Rightarrow\left(g^{2}, h, L\right)=(g, h 2, L)=1$.
Proof : Let $k \in L$. If $(g, h, k)=1$, then $g, h$ and $k$ generate a group and, clearly, $\left(g^{2}, h, k\right)=\left(g, h^{2}, k\right)=1$. So assume $(g, h, k) \neq 1$. Then $g, h$ and $g k$ cannot associate; else, they would generate a group containing $g, h$ and $k$.
So $h g^{2} \cdot k=(h g g) k$
$=(h \cdot g g) k$ by (2).
$=h(g \cdot g k)$
$=h \cdot g^{2} k$
Thus $h, g^{2}$ and $k$ associate.
Similarly $g h^{2} \cdot k=(g h h) k=(g \cdot h h) k$
$=g(h \cdot h k)$
$=g \cdot h^{2} k$
Also $g, h^{2}$ and $k$ associate.
Theorem 3.2: For any $g, h \in L, g^{2}$ and $h^{2}$ commute.
Proof: The result certainly holds if $g$ and $h$ commute or, by theorem 3.1, if $(g, h, L)=1$. So we assume that $g h \neq$ $h g$ and that $(g, h, k) \neq 1$ for some $k \in L$. There are four cases to consider.
Case 1. If no two of $g, h, k$ commute, then they are a triple of type (iv). None of these elements can commute with a product of the other two; for example, if $k$ were to commute with hg , then
$h k \cdot g=k \cdot h g=h g \cdot k=h \cdot k g$, a contradiction. Also, the square of any of $g, h, k$ associates with the other two.

To see why, suppose $g^{2}$ did not associate with $h$ and $k$. Then
$h k \cdot \mathrm{~g}^{2}=k \cdot h g^{2}$ by
$=k(h g \cdot g)$
$=(h g \cdot k) g$ by (5) and the fact that $h g$ and $k$ do not commute

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= (h\cdotkg)g since h,k,g}\mathrm{ is a type (iv) triple
=(kh\cdotg)g
= kh\cdotg}\mp@subsup{g}{}{2
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Which implies that $h k=k h$, a contradiction. Now we observe that since $h g, g, k$ do not associate and $h g$ and $g$ do not commute, $(h g \cdot g) k=g(h g \cdot k)$ by (5) and so $h g^{2} \cdot k$ $=(h g \cdot g) k=g(h g \cdot k)=g(g \cdot h k)=g^{2} \cdot h k=g^{2} h \cdot k$.
This gives $g^{2} h=h g^{2}$, so the squares of $g$ and $h$ commute as desired.
Case 2. If $g k \neq k g$ and $h k=k h$, then (because $g h \neq h g$ ) $g$, $h, k$ is a triple of type (iii) $g$. Nothing that $\left(h^{2}, k, L\right)=1$ lemma3.1, that $g h$ and $k$ commute (see theorem 2.1 (iii), and that $h(g h \cdot k)=(h g \cdot k) k$ by the right Bol identity we have,

$$
\begin{aligned}
& h^{2} g \cdot k=(h h g) k \\
& =h(h g \cdot k) \\
& =h(g h \cdot k)(h g=g h) \\
& =(h g \cdot h) k \text { by }(1) \\
& =(g h \cdot h) k \\
& =g h^{2} \cdot k
\end{aligned}
$$

And hence, again $h^{2} g=g h^{2}$ (and $g^{2}$ and $h^{2}$ commute).
Case 3. If $h k \neq k h$ and $g k=k g$, then $g, h, k$ is a triple of type (iii) $h$, so, as in case 2 , we obtain $g^{2}$ and $h^{2}$ also commute.
Case 4. If $g k=k g$ and $h k=k h$, then $g, h, k$ is a triple of type (ii) $k$. In this case $g h$ and $k$ do not commute, so $g h, h$, $k$ is a triple of type (iii) $g h$. As in case 2 , it follows that $h^{2}$ commutes with $g h$ and hence also with $g$ and with $g^{2}$.
Theorem 3.3: Let $g$ and $h$ be elements of an RA2 loop $L$. Then $\left(g^{2}, h^{2}, L\right)=1$.
Proof: Let $k \in 1$ and suppose $(g, h, K) \neq 1$. If any two of $g$, $h$ and $k$ commute, the result follows by Lemma 1. For example, if $h k=k h$, then $\left(h^{2}, g, k\right)=1$, so $h^{2}, g$ and $k$ would generate a group and $h^{2}, g^{2}$ and $k$ would associate. So we assume no two of $g, h$ and $k$ commutes. Thus they are a triple of type (IV). Now consider
$g(h \cdot g k)=(g \cdot h g) k$ by (2)
$=h g \cdot g h$
Using (5) to rewrite $(g \cdot h g) k$. (Note that the triple $g, h g, k$ does not associate and the pair $g$, $h g$ does not commute). Now $g(g \cdot k h)=g^{2} \cdot k h$. As for $h g \cdot g k$, notice that $h$ and $g k$ do not commute. (We saw this in theorem 3.2). Thus no two elements in the triple $h, g, g k$ commute; this triple is therefore of type (iv) and so $h g \cdot g k=g(h \cdot g k)=(g \cdot g k) h$ $=g^{2} k \cdot h$. We see thereby that $g^{2}, k$ and $h$ associative and the result follows.

Corollary 3.1: Let $L^{\prime}=\left\langle L^{2}\right\rangle$ be the subloop of the $R A 2$ loop $L$ generated by the squares in $l$. Then $\left(L^{\prime}, L^{\prime}, L\right)=1$.
Proof: First we use induction to show that $\left(x, h^{2}, L\right)=1$ for any $x=g_{1}{ }^{2}, g^{2}{ }_{2} \ldots . . g_{n}{ }^{2}, g_{i} \in L$, the case $n=1$ being the theorem. For $x=x_{1} g^{2}$, with $x_{1}$ the product of squares of elements in $L$, we have
$\left[x, h^{2}, k\right]=\left[x_{1} g^{2}, h^{2}, k\right]$
$=g^{2}\left[x_{1}, h^{2}, k\right]+\left[g^{2}, h^{2}, k\right] x_{1}+f\left(x_{1}, g^{2}, h^{2}, k\right)=0$
Because $f\left(x_{1}, g^{2}, h^{2}, k\right)=f\left(x_{1}, k, g^{2}, h^{2}\right)$ and $f$ vanishes whenever its last two arguments associate with all other elements. The final step, that $[x, y, k]=0$ for any $x, y \in L^{\prime}$ and $k \in L$, follows with a similar inductive argument. $\square$
The main theorem of this section is now quite straightforward.

Theorem 3.4: Let $L$ be an $R A 2$ loop. Then the sub loop $a$ generated by the squares of the elements in $L$ is an associative commutative normal sub loop of $L$.
Proof: That $L^{\prime}$ is associative follows immediately by corollary 3.1 ; that it is commutative, by theorem 3.2. To prove normality, we note that, while there are three things to check in the general setting of Bol loops, in an $R A 2$ loop we have only to verify that $k^{-1} a k \in L^{\prime}$ for every $k \in$ $L^{\prime}$ and $a \in L^{\prime}$ (see [2 Corollary 2.11]). By diassociativity, $k^{-1} g^{2} k$ is a square and then, for $x=x_{1} g^{2}$ with $x_{1}$ and $k^{-1} x_{1} k \in$ $L^{\prime}, k^{-1} x k=k^{-1}\left(x_{1} g^{2}\right) k=\left(k^{-1} x_{1} k\right)\left(k^{-1} g^{2} k\right) \in L^{\prime}$
Using the fact that $k, x_{1}$ and $g^{2}$ generate a group, by corollary 2. $\quad$.
With $L^{\prime}$ as in the theorem, we see that $L / L^{\prime}$ is an $R A 2$ loop of exponent 2 . So it is commutative and hence a group [2, corollary 2.5].

Corollary 3.2: In any $R A 2$ loop, the commutators and associators lie in the sub loop generated by the squares. If each of $a$ and $b$ is a commutator or an associator in an $R A 2$ loop, then $(a, b)=(a, b, x)=1$ for any $x \in L$. We conclude this section with a result of independent interest. It is convenient to include it here since it depends so heavily on the results just obtained. Our starting point is (7). At this stage of the proof of theorem 2 we have shown that, if $g$ and $h$ are two elements which do not commute in an RA2 loop $L$ and if $(g, h, L)=1$, then, whenever an associator $(g, a, b) \neq 1$, it is the element $k=$ $\left(g^{-1}, h^{-1}\right)$. Note that this element is independent of $a$ and $b$. Thus any associator in $L$ of the form ( $g, a, b$ ) takes on at most two values, 1 and $k$. It follows that $(g, a, k)=1$ for all $a \in L$; otherwise, $g a \cdot k=(g \cdot a k) k$ quickly gives $k=1$, a contradiction. Thus
$(g, c,(g, a, b))=1$
for all $a, b, c \in L$. We use this fact repeatedly in the next few lines. Let $a, b \in L$. Then
$g^{2} a \cdot b=(g \cdot g a) b$
$=\{g(g a \cdot b)\}(g, g a, b)$
$=\{g\{(g \cdot a b)(g, a, b)\}\}(g, g a, b)$
$=\{\{g(g \cdot a b)\}(g, a, b)\}(g, g a, b)$
$=\left(g^{2} a b\right)\{(g, a, b)(g, g a, b)\}$.
Now $g, g a$ and $b$ associate if and only if $g, a$ and $b$ do. Also, each of the associators $(g, g a, b)$ and $(g, a, b)$ assumes at most two values, 1 or $k$. It is therefore the case that these associators are equal and, because $k^{2}=1$, their product is 1 . We have shown that $g^{2}$ is in the nucleus $N$ $(L)$ and by symmetry, so is $h^{2}$. Replacing $h$ by $g h^{-1}$ in the foregoing (note that $g$ and $g h^{-1}$ do not commute and ( $g$, $\left.g h^{-1}, L\right)=1$ ), we also have $\left(g h^{-1}\right)^{2} \in N(L)$ and therefore ( $($, $h) \in N(L)$ too, because $(g, h)=g^{-2}\left(g h^{-1}\right)^{2}$. In an $R A$ loop, $(g, h, L)=1$ if and only if $(g, h)=1$ [3]. In an RA2 loop, we have the following weaker statement.

Theorem 3.5: let $L$ be an RA2 loop and $g, h \in L$. Then ( $g$, $h, L)=1 \Rightarrow(g, h) \in N(L)$. If $(g, h, L)=1$ and $(g, h) \neq 1$, then also $g^{2}, h^{2} \in N(L)$.

## Nilpotence of the augmentation ideal

A non associative ring $R$ is said to be nilpotent if, for some natural number $n$, the product of any $n$ elements in $R$, with any order of multiplication, is 0 . Defining $R^{1}=R$ and then, inductively, $R^{k+1}=R^{k} R, R$ is right nilpotent if, for some natural number $n, R^{n}=0$. In an alternative ring, right nilpotence implies nilpotence [7, p.119]. If $R$ is a ring of characteristic 2, if $l$ is an RA2 loop, and if $N$ is a normal sub loop of $L$, then the natural homomorphism $L$ $\rightarrow L / N$ extends linearly to a ring homomorphism $R L \rightarrow$ $R[L / N]$ whose kernel, denoted $\Delta(L, N)$, is the ideal of $R L$ generated by elements of the form $1+n, n \in N$. In the special case $L=N$, the homomorphism just described maps $\sum \alpha_{g} g \in R L$ to $\sum \alpha_{g} \in R$. This map, called the augmentation map, has a kernel, written $\Delta(L)$ rather than $\Delta(L, L)$, known as the augmentation ideal of $R L$. It follows directly from the definitions that, for a normal sub loop $N$ of $L, \Delta(L, N)=R L \Delta(N)$. Assume now that $R$ $=F$ is a field of characteristic2 and that the order of $L$ is $2^{n}$ for some $N>0$. Since the elements of $\ell=\{1+g / g \in L\}$ are linearly independent over $F$ and span $\Delta(L)$ (because $g(1+h)=(1+g h)+(1+g)$, it is clear $\Delta(L)$ has dimension $2^{n}-1$. As previously, we let $L^{\prime}$ be the normal sub loop of $L$ generated by the squares of $L$ and note that $L / L^{\prime}$ is a group of exponent 2 . By Jennings' result for modular group rings, $\Delta(L / L)(=\Delta(L) / \Delta(L, L)$ is nilpotent. For finite dimensional alternative rings, nilpotence is a radical property closed under extensions. Thus, to prove $\Delta(L)$ nilpotent, it suffices now to prove that $\Delta\left(L, L^{\prime}\right)$ is nilpotent. Now $\Delta(L, L\}=F L \Delta\left(L^{\prime}\right)$ is spanned over $F$ by elements of the form $g(1+a), g \in L$, $a \in L^{\prime}$ and the identity $g(1+a)=(1+g)(1+a)+(1+a)$ shows that $\Delta\left(L, L^{\prime} \subseteq \Delta(L) \Delta\left(L^{\prime}\right)=I+J I\right.$ where we have
set $I=\Delta(L)$ and $J=\Delta(L)$. Since $I$ is nilpotent (by the result for group rings), the nilpotency of $\Delta(L, L\}$ is an obvious consequence of $(I+J I)^{n} \subseteq I^{n}+I^{n} J$ for all $n \geq 1$,
a fact we proceed to establish. In so doing, we shall use regularly that $L$ and $L^{\prime}$ are commutative, that $\left(L^{\prime}, L^{\prime}, L\right)=$ 1 and hence that $\left[L^{\prime}, L^{\prime}, F L\right]=[I, I, F L]=0$. For any $g$, $h \in L$, we can write $g h=h g \cdot f_{1}$ or $h g=f_{2} \cdot g h$ for commutators $f_{1}, f_{2}$ which are in $L^{\prime}$, by corollary 3 . Then $[g, h]=g h+h g=(h g)\left(1+f_{1}\right)=\left(1+f_{2}\right)(g h)$, with both 1 $+f_{1}$ and $1+f_{2}$ in $\Delta(a)$. Furthermore, since (loop) associators of $L$ are in $L^{\prime}$, we also have such equations as $[g, h, k]=(g h \cdot k)\left(1+f_{3}\right)=\left(1+f_{4}\right)(g \cdot h k)$ with $1+f_{3}, 1$ $+f_{4} \in \Delta(L)$, whenever $g, h$ and $k$ are elements of $L$. We establish (8) with a sequence of lemmas.

Lemma 4.1: With $I=\Delta(L)$ and $J=\Delta(L), J I^{n} \subseteq I^{n}+I$ ${ }^{n} J$ for all $n \geq 1$.
Proof: An element of $J I$ is an $F$ - linear combination of elements of the form $(1+g)(1+a)$,
$g \in L, a \in L^{\prime}$. Now

$$
\begin{aligned}
& (1+g)(1+a)=(1+a)(1+g)+[(1+g),(1+a)] \\
& =(1+a)(1+g)+[g, a] \\
& =(1+a)(1+g)+g a+a g \\
& =(1+a)(1+g)+g a+a g \\
& =(1+a)(1+g)+(1+f) a g \text { for some } \\
& 1+f \in I \\
& =(1+a)(1+g)+(1+\mathrm{f}) a(1+g)+ \\
& (1+f) a,
\end{aligned}
$$

which is in $I J+I$ because $(1+f) a \in I(I$ is an ideal of $F A)$. Thus $J I \subseteq I J+I$ and the lemma is true for $n=1$. Assuming $J I^{k} \subseteq I^{k}+I^{k} J$, we have

$$
\begin{aligned}
& J I^{k+1}=J I^{k} \cdot I \subseteq\left(I^{k}+I^{k} J\right) I \\
& \subseteq I^{k+1}+I^{k} \cdot J I \\
& \subseteq I^{k+1}+I^{k}(I J+I) \\
& \subseteq I^{k+1}+I^{k+1} J+I^{k+1},
\end{aligned}
$$

from which the lemma follows. $\square$
For our next proof, it will be important to note that $I \cdot I^{n}=$ $I^{n} \cdot I\left(=I^{n+1}\right)$ for any $n \geq 1$ because elements of $I$ Icommute.

Lemma 4.2: With $I=\Delta\left(L^{\prime}\right)$ and $J=\Delta(L),\left[I^{n}, J, J\right] \subseteq I$ ${ }^{n}+$
$I^{n} J$ for all $n \geq 1$.
Proof : Let $a \in L^{\prime}$ and $g, h \in L$. Then $[1+a, 1+g, 1+h]=$ $[a, g, h]=(1+f)(a \cdot g h)$ for some $1+f \in I$. Now $(1+f)$ $(a \cdot g h)=(1+f)(1+a \cdot g h)+(1+f) \in I J+I$, so the result holds for $n=1$. Now assume $\left[I^{k}, J, J\right] \subseteq I^{k}+I^{k} J$ and let $a \in I^{k}, b \in I, x, y \in J$. Then $[a b, x, y]=b[a, x, y]+$ $[b, x, y] a$
Because $f(a, b, x, y)=f(x, y, a, b)$ and $[F L, I, I]=0$. Now
$b[a, x, y] \in I\left(I^{k}+I^{k} J\right) \subseteq I^{k+1}+I^{k+1} J$
and

$$
\begin{aligned}
& {[b, x, y] a \in(I+I J) I^{k}} \\
& \subseteq I \cdot I^{k}+I J \cdot I^{k} \\
& \subseteq I^{k+1}+I \cdot J I^{k} \\
& \subseteq I^{k+1}+I\left(I^{k}+I^{k} J\right) \text { by lemma } 4.1 \\
& \subseteq I^{k+1}+I^{k+1} J
\end{aligned}
$$

completing the induction.
Lemma 4.3: With $I=\Delta\left(L^{\prime}\right)$ and $J=\Delta(L)$, we have $(I+$ $J I)^{n} \subseteq I^{n}+I^{n} J$, for any $n \geq 1$.
Proof: When $n=1$, the result follows immediately from lemma 4.1. Assume inductively that $(I+J I)^{k} \subseteq I^{k}+I^{k} J$ for some $k \geq 1$. Then

$$
\begin{aligned}
& (I+J I)^{k+1}=(I+J I)^{k}(I+J I) \\
& \subseteq\left(I^{k}+I^{k} J\right)(I+I J) \\
& \subseteq I^{k+1}+I^{k} \cdot I J+I^{k} J \cdot I+I^{k} J \cdot J I
\end{aligned}
$$

Now $I^{k} \cdot I J=I^{k+1} J($ Since $[I, I, F L]=0)$ and $I^{k} J \cdot I=I^{k}$. $J I \subseteq I^{k}(I+I J) \subseteq I^{k+1}+I^{k+1} J$. Thus, it remains only to prove that $I^{k} J \cdot J I \subseteq I^{k+1}+I^{k+1} J$. Elements of $I^{k} J \cdot I J$ are $f$-linear combinations of elements of the form $x(1+g)$ $\cdot(1+a)(1+h)$, where $x \in I^{k}$ and where $g, h \in L$ and $a \in$ $L^{\prime}$ and such an element can be written as
$\{x(1+g)(1+a)\}(1+h)+[x(1+g), 1+a, 1+h] \ldots$ (9)
The first of these terms is $\{x(1+a)(1+g)+x[1+g, 1+$ $a]\} \cdot(1+h)=\{x(1+a)(1+g)+x[g, a]\}(1+h)$
$=\{x(1+a)(1+g)+x(1+f) a g\}(1+h)$ for some $1+f \in I$ $=\{x(1+a)(1+g)\}(1+h)+\{x(1+f)(1+a g)+x(1+$ $f)\}(1+h)$.
The last of the three here, $x(1+f)(1+h)$, is certainly in $I^{k+1} J$, while the first two are of the form $u v \cdot w$, where $u \in I^{k+1}$ and $v, w \in J$. Writing $u v \cdot w=u \cdot v w+[u, v, w]$, we see at once $u v \cdot w \in I^{k+1}+I^{k+1} J$, by the previous lemma.
The second term in (9) is
$[x+x g, 1+a, 1+h]=[x g, a, h]$
$=g[x, a, h]+[g, a, h] x+f(x, g, a, h)$
$=[g, a, h] x$

Because $\left[I^{k}, I, F L\right]=0$ and $f(x, a, g, h)=f(g, h, x, a)=0$ since $f(g, h, x, a)$ is the sum three terms involving associators in each of which two elements $x$ and $a$ (of i) appear. As for $[g, a, h] x$, we write this as $[g, h, a] x$ and observe that this can be rewritten first as $(g h \cdot a)(1+f) x$ for some
$1+f \in I$ and then as $(1+g h \cdot a)(1+f) x+(1+f) x$, which is an element of $J I^{k+1}+I^{k+1} \subseteq I^{k+1}+I^{k+1} J$ by lemma 4.1. Combining the results of this section with the known result for group rings, we have the following theorem.

Theorem 4.1: Let $F L$ be the alternative loop ring (associative or otherwise) of a loop $L$ of order $2^{n}$ over a ring of characteristic 2 . Then, with respect to any radical property for which nilpotent rings are radical and rings with 1 are not, the radical of $F L$ is its augmentation ideal $\Delta(L)$ and this is nilpotent of dimension $2^{n}-1$.

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