A Note on Characterization of Generalized Gamma Distribution by Doubly Truncated Mean Function

M. I. Khan*, M.A.R. Khan

Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh-202 002, INDIA.

*Corresponding Address:

izhar.stats@gmail.com

Research Article

Abstract: In this paper, an attempt is made to present a new characterization to generalized gamma distribution using conditional expectation in terms of failure rate and reversed failure rate. A new result is presented and some of its deductions are also discussed.

Keywords: Characterization; Conditional expectation; Generalized gamma distribution; Doubly truncated.

AMS 2000 SUBJECT CLASSIFICATION: 62G30, 62E10.

1. Introduction

The generalized gamma distribution (GGD) has been proposed by Stacy [3] as a flexible model with many applications in lifetime data analysis and several other fields and it has probability density function $(p.d.f)$ in the form.

Let X be a non-negative continuous random variable, then X has a generalized gamma Distribution (GGD) with $(p.d.f.)$

$$f(x)=\frac{\alpha}{\Gamma(\lambda)}\beta^{-\alpha\lambda}\,x^{\alpha\,\lambda-1}\exp\left(-\frac{x}{\beta}\right)^{\alpha},\ \ \alpha,\beta,\lambda>0\,;\ x>0$$

where Γ is gamma function, α,β and λ are shape, scale and index parameters respectively.

The GGD reduces to two- parameter Weibull distribution for $\lambda=1$, the two parameter gamma distribution for $\alpha=1$ and one parameter exponential distribution for $\alpha=\lambda=1$.

In the last decades thse statistical inference on GGD has been studied extensively in the literature. Stacy and Mihram [2], Parr and Webter [14], harter [5], Hager and Bain [4], Prentice [12], Lawless[6,7], Ahsanullah *et a*l. [8] and amongest others.

Characterizations based on the properties of the failure rate function have been considered by many authors. Hitha and Nair [10], Gupta and Kirmani [11], Hossain and Ahsanullah[13], Nanda [1] and Nofal [9].

Let X be a random variable (r.v.) usually representing the life length for a certain unit, then the r.v. $(X-x\,|\,X\geq x)$, represents the residual life of a unit with age x. The failure rate or (hazard) function, defined by

$$h(x)=\frac{f(x)}{1-F(x)}\ ,\ 0\leq x<\infty$$

represents the failure rate of X at the age x where $F(x)=P[X\leq x]$ and $f(x)$ is $(p.d.f.)$.

The problem of characterizations of distributions are today a substantial part of probability theory and mathematical statistics. The mean residual life is applicable in biostatistics and many other actuarial sciences, engineering, economics, biometry, applied probability areas and developing various criteria for ageing. They also are useful in survival analysis.

2. Notations

$$h(x)=\frac{f(x)}{\overline{F}(x)},\quad r(x)=\frac{f(x)}{F(x)}$$

$$\frac{h(x)}{r(x)}=\frac{F(x)}{\overline{F}(x)}\ ,\quad \frac{r(x)}{h(x)}=\frac{\overline{F}(x)}{F(x)}$$

Before coming to the main results, some Lemmas are given which is used in Theorem.

3. Lemmas

Lemma-3.1:

$$F(y)-F(x)=F(y)\,\overline{F}(x)-F(x)\,\overline{F}(y)$$

Lemma-3.2:

$$\frac{F(y)-F(x)}{\overline{F}(y)\overline{F}(x)}=\left[\frac{h(y)r(x)-r(y)h(x)}{r(x)\,r(y)}\right]$$

Lemma-3.3:

$$\frac{xf(x)-yf(y)}{\overline{F}(y)\overline{F}(x)}=\left[\frac{xh(x)r(x)r(y)+xh(x)h(y)r(x)-yh(y)r(x)\,r(y)-yh(x)h(y)r(y)}{r(x)\,r(y)}\right]$$

Lemmas can be proved in view of Nofal [9].

4. Characterization Theorem

Theorem 4.1: Let X be non- negative continuous random variable then X has a generalized gamma distribution with probability density function(*p.d.f.*)

$$f(x)=\frac{\alpha}{\Gamma(\lambda)}\beta^{-\alpha\lambda}\,x^{\alpha\lambda-1}\exp\left(-\frac{x}{\beta}\right)^{\alpha},\ \alpha,\beta,\lambda>0,\ x>0 \tag{4.1}$$

If and only if

$$E[X \mid x\le X\le y]=\beta\left[\lambda+\frac{1}{\alpha}\frac{xh(x)r(x)r(y)+xh(x)h(y)r(x)-yh(y)r(x)\,r(y)-yh(x)h(y)r(y)}{h(y)r(x)-h(x)r(y)}\right] \tag{4.2}$$

Proof: To prove the necessary part, we know that,

$$E[X \mid x\le X\le y]=\int_x^y\frac{u\,f(u)}{[F(y)-F(x)]}du$$

$$=\frac{1}{[F(y)-F(x)]}\left[\int_x^y u\frac{\alpha}{\Gamma\lambda}\beta^{-\alpha\lambda}\,u^{\alpha\lambda-1}\exp\left(-\frac{u}{\beta}\right)^{\alpha}du\right] \tag{4.3}$$

Integrating (4.3) by parts ,then

$$E[X \mid x\le X\le y]=\frac{\beta}{\alpha[F(y)-F(x)]}[-yf(y)+x\,f(x)]+\alpha\lambda[F(y)+F(x)] \tag{4.4}$$

Using Lemmas 3.1, 3.2 and 3.3, we have

$$E[X \mid x\le X\le y]=\beta\left[\lambda+\frac{1}{\alpha}\frac{xh(x)r(x)r(y)+xh(x)h(y)r(x)-yh(y)r(x)\,r(y)-yh(x)h(y)r(y)}{h(y)r(x)-h(x)r(y)}\right] \tag{4.5}$$

This proves the necessary part. To prove the sufficiency part (4.4) can be written as,

$$\int_x^y u\,f(u)du=-\frac{\beta}{\alpha}\,y\,f(y)+\frac{\beta}{\alpha}\,x\,f(x)+\beta\,\lambda F(y)-\beta\,\lambda F(x) \tag{4.6}$$

Differentiating (4.6) both sides with respect to y

$$\frac{f'(y)}{f(y)}=-\frac{\alpha}{\beta}+\frac{(\alpha\lambda-1)}{y} \tag{4.7}$$

Integrating (4.7) both side both sides with respect to y

$$\log f(y)=-\frac{y}{\beta}\alpha+\log y^{\alpha\lambda-1}$$

$$f(y)=k\,y^{\alpha\lambda-1}\exp\left(-\frac{y}{\beta}\right)^{\alpha} \tag{4.8}$$

Using the fact that $\int_0^\infty f(y)dy = 1$, then $k = \frac{\alpha}{\Gamma(\lambda)} \beta^{-\alpha\lambda}$

Putting the value of k in (4.8) , we have

$$f(x) = \frac{\alpha}{\Gamma(\lambda)} \beta^{-\alpha\lambda} x^{\alpha\lambda - 1} \exp\left(-\frac{x}{\beta}\right)^{\alpha}, \quad \alpha, \beta, \lambda > 0, \; x > 0$$

Which is generaliged gamma distribution.

Remark : Putting $\alpha = 1$ in Theorem 4.1, we get the result as obtained by Nofal, M. Z. (2011).

References

1. A. K. Nanda. (2010). Characterization of distributions through failure rate and mean residual life functions. *Statistical Probability Letters*, 80, 752-755.
2. E. Stacy and G. Mihram. (1965). Parameter estimation for generalized gamma distribution. *Technometrics*,7, 349-358.
3. E. Stacy. (1962). A generalization of the gamma distribution. *Annals of Mathematical Statistics*, 1187-1192.
4. H. Hager and L. Bain. (1970). Inferential procedure for the generalized gamma distribution. *Journal of American Statistical Association*, 65, 1601-1609.
5. H. L. Harter. (1967). Maximum likelihood estimation of the parameter of a four-parameter generalized gamma population from complete censored samples. *Technometrics*, 9, 159-165.
6. J. Lawless. (1980). Inference on the generalized gamma and log gamma distributions. *Technometrics*, 22, 409-419.
7. J. Lawless. (1982). *Statistical Models and Method for Lifetime data*. John Wiley and sons, New York.
8. M. Ahsanullah, M. Maswadah, M and M. Seham .(2011) . Statistical inference on the generalized gamma distribution based on generalized order statistics. *Journal of Applied Statistical Sciences*,19(4),159-167.
9. M. Z. Nofal. (2011). Characterization of beta and gamma distribution based on doubly truncated mean function. *Journal of Applied Statistical Sciences*, 19(2), 159-167.
10. N. Hitha and N. U. Nair. (1989). Characterization of some models by properties of residual life. *Calcutta Statistical Association Bulletin*, 38, 219-225.
11. R. C. Gupta and S.N.U.A. Kirmani. (2004). Some characterization of distributions by functions of failure rates and mean residual life. *Communication in Statistics Theory and Method*, 33(12), 3115-3131.
12. R. Prentice. (1974). A log gamma model and its maximum likelihood estimation. *Biometrika*, 61, 539-544.
13. S. A. Hossain and M. Ahsanullah. (2010). Mean residual lives of some discrete distributions *Pakistan journal of Statistics*, 26(4), 565-568.
14. V. B. Parr and J. J.Webster. (1965). A method for discriminating between failure density functions used in reliability predictions. *Technometrics*, 7, 1-10.