Rings with \((x, R, x)\) in the Left Nucleus

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**Abstract**

If \(N_i\) and \(N_j\) be the Lie ideals of a nonassociative ring \(R\), then \([N_i, R] \subseteq N_i\) and \([N_j, R] \subseteq N_j\). Also if \((x, R, x)\) is in the left nucleus then \(N_i[R, R] \subseteq N_i\). If \(R\) is a prime ring with \(N_i \neq 0\), and \((x, R, x)\) in the left nucleus then \(R\) is either associative or commutative.

**Key Word:** Nonassociative ring, Left nucleus, Right nucleus, Lie ideals, Associator ideal.

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**INTRODUCTION**

Kleinfeld [1] studied nonassociative rings with \((x, R, x)\) and \([R, R]\) in the left nucleus. Yen [2] considered the rings with the weaker hypothesis that is, rings with \((x, R, x)\) and \([N_i, R]\) in the left nucleus and proved that if \(R\) is a semiprime ring, then \(N_i = N_j\). He also proved that if \(R\) is a prime ring with \(N_i \neq 0\) satisfying one additional condition \(N_i[R, R] \subseteq N_i\), then \(R\) is either associative or commutative. In this paper by considering \(N_i\) and \(N_j\) as the Lie ideals of a ring \(R\), we present some properties of \(R\) with \((x, R, x)\) in the left nucleus. Using these properties, we show that \(N_i[R, R] \subseteq N_i\). Also we prove that, if \(R\) is a prime ring with \(N_i \neq 0\), then \(R\) is either associative or commutative.

**PRILIMENARIES**

In a nonassociative ring \(R\) we define an associator as \((x, y, z) = (xy)z - x(yz)\) and the commutator as \([x, y] = xy - yx\) for all \(x, y, z \in R\). To make the notation more convenient we often use \(\cdot\) to indicate multiplication as well as juxtaposition. In products, juxtaposition takes precedence, i.e., \(xy \cdot z = (xy)z\). The nucleus of a ring \(R\) is defined as \(N = \{n \in R / (n, R, R) = (R, n, R) = (R, R, n) = 0\}\), the right nucleus as \(N_r = \{n \in R / (R, R, n) = 0\}\) and the left nucleus as \(N_l = \{n \in R / (n, R, R) = 0\}\). A ring \(R\) is said to be prime if whenever \(A\) and \(B\) are ideals of \(R\) such that \(AB = 0\), then either \(A = 0\) or \(B = 0\) and is said to be semiprime if for any ideal \(A\) of \(R\), \(A^2 = 0\) implies \(A = 0\). These rings are also referred to as rings free from trivial ideals. And a ring is said to be simple if whenever \(A\) is an ideal of \(R\), then either \(A = R\) or \(A = 0\).

Let \(R\) be a nonassociative ring satisfying \((x, R, x) \subseteq N_i\), that is,

\[(x, y, z) + (z, y, x) \in N_i\]  \hspace{1cm} (1)

Let \(N_i\) and \(N_j\) be the Lie ideals of \(R\). Then

\([N_i, R] \subseteq N_i\]  \hspace{1cm} (2)

\([N_j, R] \subseteq N_j\]
We use Teichmuller identity which is valid in any arbitrary ring.

\[(wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y) z = 0, \quad (3)\]

for all, \( w, x, y, z \in R \).

Then with \( w = n \in N_l \) in (3), we obtain

\[(nx, y, z) = n(x, y, z). \quad (4)\]

Since \( N_l \) is the Lie ideal from (2), we obtain

\[(nx, y, z) = n(x, y, z) = (xn, y, z), \quad (4)\]

for all, \( n \in N_l \).

Thus \( N_l \) is the associative subring of \( R \).

**MAIN SECTION**

**Lemma 3.1:** Let \( T = \{ t \in N_l; t(R, R, R) = 0 \} \), then \( T \) is an ideal of \( R \).

**Proof:** In (4) substituting \( n = t \), we obtain

\[(tx, y, z) = t(x, y, z) = (xt, y, z) = 0. \quad (5)\]

Thus \( tR \subset N_l \) and \( Rt \subset N_l \).

Also, \( tw \cdot (x, y, z) = t \cdot w(x, y, z) \).

Multiplying (3) with \( t \) on the left side, we obtain

\[ t \cdot w(x, y, z) = -t \cdot (w, x, y)z \]

\[ = -t (w, x, y) \cdot z \]

\[ = 0. \]

Hence \( tw \cdot (x, y, z) = 0 \). Thus \( TR \subseteq T \).

Now using \( TR \subseteq T \), (2), (4), \( RT \subset N_l \) and (1), we obtain

\[ wt \cdot (x, y, z) = [w, t] (x, y, z) \]

\[ = ([w, t]x, y, z) \]

\[ = ((wt)x, y, z) - ((tw)x, y, z) \]

\[ = ((wt), y, z) + (x(wt), y, z) - (t(wx), y, z) \]

\[ = ([wt], y, z) + (x(wt), y, z) \]

\[ = -((x, w, t), y, z) + ((xw)t, y, z) \]

\[ = -((x, w, t) + (t, w, x), y, z) \]

\[ = 0. \]

Hence \( RT \subseteq T \). Thus \( T \) is an ideal of \( R \). From the definition of \( T \), we obtain \( T(R, R, R) = 0 \).

This completes the proof of the Lemma.

Let \( A \) be the associator ideal of \( R \). We assume that \( R \) satisfies (1) and also \( R \) is semiprime. Using Lemma 3.1 and equation (3), we obtain \( T \cdot A = 0 \) and hence \( (T \cap A)^2 = 0 \). Thus we have \( T \cap A = 0 \) and \( A \cdot T = 0 \).

From Lemma 3.1 and equation (3), we obtain

\[(R, T, R) = 0. \quad (6)\]

**Lemma 3.2:** Let \( R \) be a nonassociative ring satisfying \((x, y, z) + (z, y, x) \in N_l \). Then \((R, R, N_l) = 0 \).

**Proof:** Let \( n \in N_l \), then from (1), we obtain

\[(x, y, n) = (x, y, n) + (n, y, x) \in N_l. \]

Also from (3), we obtain

\[z (x, y, n) = (zx, y, n) - (z, xy, n) + (z, x, yn) - (z, x, y)n.\]

Hence using these, (4) and (1), we obtain

\[(x, y, n)(z, r, s) = (zx, y, n), r, s \]

\[= ((zx, y, n), r, s) - ((z, xy, n), r, s) + ((z, x, yn), r, s) - ((z, x, y)n, r, s) \]

\[= (z, x, yn), r, s) - ((z, x, y)n, r, s) \]

\[= -((yn, x, z), r, s) - (n(z, x, y)), r, s) \]

\[= -((n(y, x, z), r, s) - n((z, x, y), r, s) \]

\[= -n((y, x, z), r, s) - n((z, x, y), r, s) \]

\[= -n((y, x, z) + (z, x, y), r, s) \]

\[= 0. \]

Hence \((x, y, n) \in T. \)
Since \((x, y, n)\) is also an associator, it is also in \(A\).

Thus from (5), we obtain \((x, y, n) = 0\).

Hence \((R, R, N_r) = 0\).

From Lemma 3.2, we obtain \(N_l \subseteq N_r\).

Let \(n \in N_r\). Then with \(z = n\) in (3), we obtain 
\[(w, x, yn) = (w, x, y)n \text{ for all } n \in N_r \text{ and } w, x, y \in R.\]  \hfill (8)

**Lemma 3.3:** Let \(N_l\) be the Lie ideal of \(R\) and let 
\[S = \{n \in N_r: (R, R, R)n = 0\}, \text{ then } S \text{ is an ideal of } R, (R, R, R)S = 0, S \cap A = 0, S \cdot A = A \cdot S = 0 \text{ and } T \subseteq S.\]

**Proof:** Using (1), (3), (5), (7) and (8) and the proof of Lemma 3.1, this Lemma is proved.

**Lemma 3.4:** If \(N_l\) and \(N_r\) are the Lie ideals of \(R\), then \(N_r = N_l\) and \(S = T\).

**Proof:** Let us assume that \((R, R, n) = 0\), then from (1), we obtain 
\[(n, x, y) = (n, y, x) + (y, x, n) \in N_r.\]

Now using (1), (7), (8) and \([N_r, R] \subseteq N_r\) and since \(N_r\) is an associative subring of \(R\), we obtain 
\[(n, y, z) - n(x, y, z) = \{(nx, y, z) + (z, y, nx)\} - n((x, y, z) + (z, y, x)) + [n, (z, y, x)] \in N_r.\]

From the above equation and \((n, R, R) \subseteq N_l \subseteq N_r\) and with \(w = n\) in (3), we obtain 
\[(n, x, y)z = \{(nx, y, z) - n(x, y, z)\} - n(xy, z) + (n, x, yz) \in N_r.\]

Hence using this and (8), we obtain 
\[(s, r, z) (n, x, y) = (s, r, (n, x, y)z) = 0, \text{ which shows that } (n, x, y) \in S \cap A \text{ and thus from } 3.3, \text{ we have } (n, x, y) = 0.\]

Hence \(N_r \subseteq N_l\). Thus from (7), we have \(N_r = N_l\). From Lemma 3.3 again, \(S \cdot A = 0\) and so \(S = T\). This completes the proof of the Lemma.

**Theorem 3.1:** If \(R\) is a semiprime ring satisfying \((x, y, z) + (z, y, x) \in N_r\), where \(N_r\) is the Lie ideal of \(R\), then \(T\) is an ideal of \(R\) and \((N_r, R) = R, (R, R)N_r = 0\). Also, if \([N_r, R] \subseteq N_r\), then \(N_r = N_l\) and \(S = T \subseteq N\).

**Proof:** From (6) and Lemmas 3.1, 3.2, 3.3 and 3.4 the Theorem is proved.

**Lemma 2.5:** Let \(I = \{a \in R : N_l \cdot a = 0\}\), then \(I\) is an ideal of \(R\).

**Proof:** First we show that \((R, R, R) \subseteq I\). By taking \(y = z = x\) in (1), we obtain 
\[(x, x, x) + (x, x, x) = 2(x, x, x) \in N_l.\]

So \((x, x, x) \in N_l\).

Let \(S(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)\).

Now linearization of \((x, x, x)\) gives \((x, y, z) + (y, z, x) + (z, x, y) + (x, y, z) + (z, y, x) + (x, z, y) + (x, z, y) \in N_l.\)

i.e., \(S(x, y, z) + S(y, x, z) \in N_l\).

We have \(D(x, y, z) = [xy, z] - x[y, z] - [x, z]y - (x, y, z) - (z, x, y) + (z, x, y) = 0.\)

This identity is valid in any arbitrary ring.

Now \(D(x, y, z) - D(y, x, z)\) gives 
\[[x, y, z] + [y, z, x] + [z, x, y] = S(x, y, z) - S(y, x, z).\]

If \(z \in N_l\), we obtain \(S(x, y, z) \in N_l.\)

But from (9), \(S(x, y, z) \in N_l.\)

i.e., \(2S(x, y, z) \in N_l.\)

i.e., \(S(x, y, z) \in N_l.\)

i.e., \((x, y, z) + (y, z, x) + (z, x, y) \in N_l.\)

But \((x, x, x) \in N_l\) implies \((x, x, x) \in N_l.\)

i.e., \((R, N_l, R) \subseteq N_l.\) implies \((R, N_l, R) = (R, N_l, R) = 0.\)

Now in (10) substituting \(x = n\) and forming the associators with \(r, s\) and using (12), we obtain 
\[(ln, y, z, r, s) + ((n, y, z) r, s) + ((n, z, y) r, s) + ((z, n, y) r, s) - (n, z, y, r, s) = (n, y, z) r, s) + ((n, z, y) r, s) + ((z, n, y) r, s).


i.e., \((N_l[R, R], R, R) = ([N_l[R, R], R, R] - ([N_l, R, R], R, R)\)
Thus \( N_i([R, R], R, R) = (N_iR, R, R) = 0 \) from (12).

Now let \( a \in I, n \in N_i \) and \( x, y, z, w \in R \). Thus we obtain
\[
n(ax) = (na)x = 0 \text{ implies } IR \subseteq I.
\]

Now from (13), we obtain
\[
n(xa) = n[x, a] \in N_i.
\]

Since \( na = 0 \) and \( n \in N_i \), we obtain \( n(a, x, y) = 0 \).

Using (15), (1) and since \( N_i \) is an associative subring of \( R \), we obtain
\[
n((y)x)a - n(y(xa)) = n(y, x, a)
\]
\[
= n((a, x, y) + (y, x, a)) \in N_i.
\]

Applying (16) and \( n(xa) \in N_i \), we obtain
\[
n(y(xa)) \in N_i
\]

Using (17) and (13), we obtain
\[
(n(xa))y = n((xa)y)
\]
\[
= n[xa, y] + n(y(xa)) \in N_i.
\]

Combining the above with \( n(xa) \in N_i \), we obtain
\[
n(xa)(y, z, w) = ((n(xa))y, z, w)
\]
\[
= 0.
\]

Hence \( n(xa) \in T \) and thus \( n(xa) = 0 \) implies \( RI \subseteq I \).

Therefore \( I \) is an ideal of \( R \) and thus \( NI = 0 \).

**Theorem 3.2:** If \( N_i \) is the Lie ideal of a prime ring \( R \) with \( N_i \neq 0 \) and satisfying \((y, x, z) + (z, y, x) \in N_i \), then \( R \) is either associative or commutative.

**Proof:** Since \( R \) is prime using (5), we obtain either \( A = 0 \) or \( T = 0 \). If \( A = 0 \), then \( R \) is associative. Hence we assume that \( T = 0 \). Since \( N_i \) is the Lie ideal of \( R \), using Lemma 3.2, we see that the ideal of \( R \) generated by \( N_i \) is \( N_i + N_iR \). Then \( NI = 0 \) from Lemma 3.5. Hence we obtain
\[
(N_i + N_iR)I \subseteq N_iI + (N_iR)I
\]
\[
= N_iI + (N_i, R, I) + N_i(RI)
\]
\[
\subseteq N_iI + N_i(RI)
\]
\[
= 0.
\]

Thus \( [R, R] \subseteq N_i \). Now \( R \) satisfies Kleinfeld’s hypothesis [1]. Hence it follows that \( R \) is either associative or commutative. This completes the proof of the Theorem.

**REFERENCES**