Research Article

Modeling of one dimensional unimodal map to its corresponding networks

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Abstract

Introduction: Here we consider the unimodal map of dimension one i.e. logistic map. We search the parameter values for which the model undergoes period doubling bifurcation. The set of periodic points at the nth bifurcation point if transformed suitably serves as the vertices set V_n .The set of edges on the set V_n is defined to form horizontal visibility graph and its modified one which has become famous lately in the literature, and is denoted as $E_{m,n}$.Some properties of the graph $G_{m,n} = (V_n, E_{m,n})$ is studied in this paper.
Keywords: Period- Doubling Bifurcation/ Periodic Points / Horizontal Visibility Graph/2010 AMS
Classification: 37 G 15, 37 G 35, 37 C 45, 05 C 07

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Received Date: 23/10/2014 Accepted Date: 05/11/2004

Access this article online	
Quick Response Code:	Website: www.statperson.com
	DOI: 02 November 2014

INTRODUCTION

Since the discovery of different dynamical aspects of logistic map by Robert May[1], it has become a role model for showing the existence of chaos and related properties in one dimensional system due to its simple expressions yet complicated behavior. The difference equation for logistic model is given by $x_{n+1} = ax_n(1 - x_n)$ where a is the control parameter. Feigenbaum showed that the model follows a period doubling root to chaos which obeys a universal rule. Since then difference aspect of dynamical properties of logistic map have been studied by different scientist and engineers. One of the aspects of study the dynamical behavior is to map time series into graphs as it has become attractive in recent days due to its combination of two giant fields of modern science as dynamical system and complex networks theory. Zhang and Small[6] developed a method that mapped each cycle of a pseudo periodic time series into a node in a graph. Xuetal[5] concentrated in the relative frequencies of appearance of four –node motifs inside a particular graph in order to classify it into a particular superfamily of networks which corresponded to specific underlying dynamics of the mapped time series.

How to site this article: Tarini Kumar Dutta, Debasmriti Bhattacherjee, Debasish Bhattacharjee. Modeling of one dimensional unimodal map to its corresponding networks. *International Journal of Statistika and Mathemtika* Nov 2014 to Jan 2015; 12(1): 55-58. http://www.statperson.com (accessed 05 November 2014).

CONSTRUCTION OF THE VERTICES OF THE HVG FOR UNIMODAL MAP

As we know that logistic map exhibits period doubling scenario to form chaos, we consider the periodic points at the parameters where bifurcation occurs, to construct the vertices set of the HVG. The procedure of forming vertices is as follows:

Let $\{x_1, x_2, \dots, x_n\}$ be n periodic points at some bifurcation point.Let $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$, where $x_{i_1}, x_{i_2}, \dots, x_{i_n}$ be the rearrangement of the terms of x_i , i = 1,2 ,…,n. Then the vertices set at that bifurcation point is $\{x_{i,j} |$ith periodic point has j position in increasing order in n periodic points$\}$ Based on the above construction, different sets of vertices at different bifurcation points are calculated from a computer

Program and some of them are given as follows:

$V_1 = \{x_{0,0}, x_{1,1}\}$

$V_2 = \{x_{0,0}, x_{2,1}, x_{1,2}, x_{3,3}\}$

$V_3 = \{x_{0,0}, x_{4,1}, x_{6,2}, x_{2,3}, x_{1,4}, x_{5,5}, x_{3,6}, x_{7,7}\}$

In this paper a particular class of vertex set has been taken where the edge set definition have been taken a modified form of horizontal visibility algorithm. The objective is to achieve the graph theoretic properties of the class. The second section of the paper represents properties mainly on degrees of the vertex class while the first section gives some propositions which are the basis of the second section.

Let $V_1 = \{x_{0,0}, x_{1,1}\}$.Let V_{n-1} be the set containing 2^{n-1}elements then we consider the following collections:

$V'_{n,1} = \{x_{2k,i} |\ x_{k,i} \in V_{n-1}, i = 0,1,2, \dots, 2^{n-2} - 1\}$

$V'_{n,2} = \{x_{2k+4,i} | x_{k,i} \in V_{n-1}, i = 2^{n-2}, \dots, 2^{n-1} - 2, and, n > 2\}$

$V'_{n,3} = \{x_{2k+4-2^n, 2^{n-1}-1} | x_{k,2^{n-1}-1} \in V_{n-1}\}$

$V'_{n,4} = \{x_{2k+1, 2^{n-1}+i}, x_{k,i} \in V_{n-1}, i = 0,1,2, \dots, 2^{n-1} - 1\}$

Then $V_n = \cup_{i=1}^{4} V'_{n,i}$

Let$E_{m,n} = \{(x_{n_1,i}, x_{n_2,j}) | N(x_k, x_{k_1}) \leq$ m for , $k_1 \geq$ i, or j and, $n_1 < k < n_2\}$

where $N(x_k, x_{k_1})$ represents the number of elements and m is a non negative integers.

Clearly

$E_{0,n} \subseteq E_{1,n} \subseteq E_{2,n} \subseteq E_{3,n} \subseteq \dots$

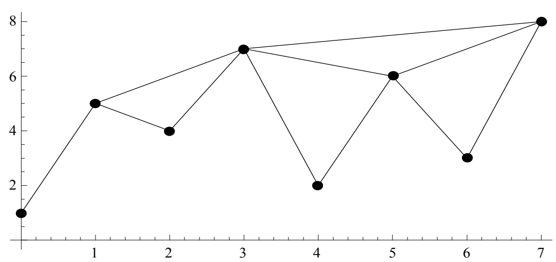

Figure 2.1: represents the graph (V_3,$E_{0,3}$) the point $x_{i,j}$ in the co-ordinate (i, j+1)

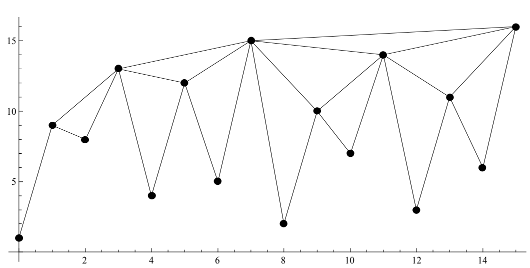

Figure 2.2: represents the graph (V_4,$E_{0,4}$)the point $x_{i,j}$ in the co-ordinate (i, j+1)

Theorem 2.1: For all n ≥ 1.The set V_n always contain one element $x_{2^n-1, 2^n-1}$.

Proof: First we consider the set$V_1 = \{x_{0,0}, x_{1,1}\}$. Clearly,

$V_2 = \{x_{0,0}, x_{1,2}, x_{2,1}, x_{3,3}\}$.Also, $V_3 = \{x_{0,0}, x_{4,1}, x_{6,2}, x_{2,3}, x_{1,4}, x_{5,5}, x_{3,6}, x_{7,7}\}$.

Clearly the result is true for n=1,2,3.Let the result be true for V_{n-1}i.e. it contains$x_{2^{n-1}-1, 2^{n-1}-1}$.From the construction of$V'_{n,4}$,we have for i=$2^{n-1} - 1$,the element $x_{2.(2^{n-1}-1)+1, 2^{n-1}+2^{n-1}-1} = x_{2^n-1, 2^n-1}$.

Theorem 2.2: The set V_n contains the element $x_{2,2^{n-1}-1}$ for all V_n, for $n \geq 2$.
Proof: First we consider the set $V_1 = \{x_{0,0}, x_{1,1}\}$ .Clearly,
$V_2 = \{x_{0,0}, x_{1,2}, x_{2,1}, x_{3,3}\}$.Also, $V_3 = \{x_{0,0}, x_{4,1}, x_{6,2}, x_{2,3}, x_{1,4}, x_{5,5}, x_{3,6}, x_{7,7}\}$. Hence the result is true for n=2,3.We consider the element $x_{2^{n-1}-1,2^{n-1}-1}$ for the set V_{n-1}.Now we consider the construction of $V'_{n,3}$.The element $x_{2^{n-1}-1,2^{n-1}-1}$ will be transformed to the form $x_{2(2^{n-1}-1)+4-2^n,2^{n-1}-1}$ i.e. $x_{2,2^{n-1}-1}$.
Theorem 2.3: For all $n \geq 1$ the set V_n contains $x_{1,2^{n-1}}$.
Proof: As we see that $x_{0,0}$ is an element for V_n for all $n \geq 1$.From the construction of $V'_{n,4}$, the element of $x_{0,0}$ of V_{n-1} will be transformed to $x_{1,2^{n-1}}$.
Theorem 2.4: The elements of the form $x_{1+2+2^2+2^3+\ldots+2^k,2^{n-1}+2^{n-2}+\ldots 2^{n-k-1}}$ exists.
Proof: Already we know that $x_{0,0}, x_{1,2^{n-2}} \in V_{n-1}$ .From the construction of $V'_{n,4}$ we have $x_{2.1+1,2^{n-1}+2^{n-2}}$.Now we consider the element $x_{2.1+1,2^{n-2}+2^{n-3}}$ in V_{n-1}.Again from the construction of $V'_{n,4}$, we have $x_{2(2.1+1+1),2^{n-1}+2^{n-2}+2^{n-3}}$ i.e $x_{1+2+2^2,2^{n-1}+2^{n-2}+2^{n-3}}$. We can repeat the above process and show that the element $x_{1+2+2^2+2^3+\ldots+2^k,2^{n-1}+2^{n-2}+\ldots 2^{n-k-1}}$ exists provided $n \geq k+1$.
Theorem 2.5: If $x_{k,i}$ exists where $2^{n-1} \leq i < 2^{n-1} + 2^{n-2}$ then k=1+4m, $m = 0,1,2,3,\ldots,2^{n-2}-1$ for V_n.
Proof: We consider the set V_{n-1}.This set contains 2^{n-1} elements.It contains the elements of the form $x_{2m,i}, i = 0,1,\ldots,2^{n-2}-1$ and $m = 0,1,2,\ldots,2^{n-2}-1$.From the construction of $V'_{n,4}$.the elements become $x_{2.2m+1,2^{n-1}+i}$, i.e. $x_{4m+1,2^{n-1}+i}, i = 0,1,\ldots,2^{n-2}-1$ and $m = 0,1,2,\ldots,2^{n-2}-1$.
Theorem 2.6: If $x_{k,i}$ exists where $2^{n-1} + 2^{n-2} \leq i < 2^{n-1} + 2^{n-2} + 2^{n-3}$,then k=3+8m, $m = 0,1,2,3,\ldots,2^{n-3}-1$ for V_n.
Proof: We consider the set V_{n-2}.This set contains 2^{n-2} elements.It contains the elements of the form $x_{2m,i}, i = 0,1,\ldots,2^{n-3}-1$ and $m = 0,1,2,\ldots,2^{n-3}-1$.From the construction of $V'_{n-1,4}$.The elements become $x_{2.2m+1,2^{n-2}+i}$, i.e. $x_{4m+1,2^{n-2}+i}, i = 0,1,\ldots,2^{n-3}-1$ and $m = 0,1,2,\ldots,2^{n-3}-1$.Again from the construction of $V'_{n,4}$ the elements become $x_{2(4m+1)+1,2^{n-1}+2^{n-2}+i}$,i.e. $x_{8m+3,2^{n-1}+2^{n-2}+i}, i = 0,1,\ldots,2^{n-3}-1$ and $m = 0,1,2,\ldots,2^{n-3}-1$. Similarly starting with the elements of V_{n-3} , considering the elements of the form $x_{2m,i}, i = 0,1,\ldots,2^{n-4}-1$ and $m = 0,1,2,\ldots,2^{n-4}-1$ it can be established that if $x_{k,i}$ exists where $2^{n-1} + 2^{n-2} + 2^{n-3} \leq i < 2^{n-1} + 2^{n-2} + 2^{n-3} + 2^{n-4}$,then k=5+16m , $m = 0,1,2,3,\ldots,2^{n-4}-1$ for V_n. On repeating the above process we may get the following theorem:
Theorem 2.7: If $x_{k,i}$ exists where $\sum_{t=1}^{r} 2^{n-t} \leq i < \sum_{t=1}^{r+1} 2^{n-t}$, then $k = \sum_{t=0}^{r-1} 2^t + 2^{r+1}m$, $m = 0,1,2,3,\ldots,2^{n-r-1}-1$ for V_n.
Theorem 2.8: The element $(x_{1,2^{n-1}}, x_{1+2,2^{n-1}+2^{n-2}}) \in E_{0,n}$ for all $n \geq 2$.
Proof: For this we want to show that for any $x_{k,i}$ if $2^{n-1} < i < 2^{n-1} + 2^{n-2}$, then k>3.It is clear that we have to see the value of i for which $x_{2,i}$ exists.From the theorem 2 we see that $x_{2,2^{n-1}-1}$ exists, so it is clear that if k=2, $i < 2^{n-1}$.Hence the definition of $E_{0,n}$ implies $(x_{1,2^{n-1}}, x_{1+2,2^{n-1}+2^{n-2}}) \in E_{0,n}$ In fact we can generalize the above theorem and conclude as follows:
Theorem 2.9: The element $(x_{\sum_{i=0}^{i=k-1} 2^i, \sum_{i=n-1}^{i=n-k+1} 2^i}, x_{\sum_{i=0}^{i=k} 2^i, \sum_{i=n-1}^{i=n-k} 2^i}) \in E_{0,n}$ for all $n \geq 2$.
Proof: The idea is similar to theorem 8 i.e. if $2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^{n-k+1} < i < 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^{n-k}$ then to show that $k > 1 + 2 + 2^2 + \ldots 2^k$ which is obvious from theorem 7.

DEGREES OF VARIOUS VERTICES OF THE SET V_n

Theorem 3.1: The degree of the point $x_{2^n-1,2^n-1}$ for the set V_n is n.
Proof: The idea of the proof is to search the elements $x_{k,i}$ such that for x_{k_1,i_1} if $k < k_1, i > i_1 \; \forall \; k_1, i_1$. Clearly from the definition of $E_{0,n}$ $x_{2^n-1,2^n-1}$ is attached to its immediate point $x_{2^n-2,k}$ for some k. Now we consider the elements $x_{1,2^{n-1}}, x_{1+2,2^{n-1}+2^{n-2}}$ All the elements between them are of the form $x_{4m+1,i}$, $2^{n-1} < i < 2^{n-1} + 2^{n-2}$.We consider the element $x_{2^n-3,k}$,for some k.As $2^n - 3 = 4m + 1$,for some m,k lies between 2^{n-1} and $2^{n-1} + 2^{n-2}$.Similarly, if we consider the elements $x_{k,i}$, where $2^{n-1} + 2^{n-2} < i < 2^{n-1} + 2^{n-2} + 2^{n-3}$.The elements are of the form $x_{8m+3,i}$.Hence it can be easily seen that $8m + 3 = 2^n - 5$,for some m and $16m + 7 = 2^n - 9$,for some m and so on. Hence we can make partition of the set $V_n \setminus \{x_{2^n-1,2^n-1}\}$ in the following form

$V_{n,1} = \{x_{k,i} \,|k = 2m, 0 < m < 2^{n-1} - 1, 0 < i < 2^{n-1} - 1\}$
$V_{n,2} = \{x_{k,i}|k = 4m + 1,0 < m < 2^{n-2} - 1, 2^{n-1} < i < 2^{n-1} + 2^{n-2} - 1\}$
$V_{n,3} = \{x_{k,i}|k = 8m + 3,0 < m < 2^{n-3} - 1,2^{n-1} + 2^{n-2} < i < 2^{n-1} + 2^{n-2} + 2^{n-3} - 1\}$
$V_{n,n} = \{x_{k,i}|k = 2^n\, m + \sum_{t=0}^{n-2} 2^t\,, m = 0, i = 2^n - 2\}$
Exactly one element $x_{2^j(2^{n-j}-1),i}$for some i of each set $V_{n,j}$is attached with the element$x_{2^n-1,2^n-1}$.Hence the degree of the said element is n.Hence the theorem.

Lemma 3.2: The graph $G_{0,n}$ =($V_n, E_{0,n}$) is always a subgraph of $G_{0,n+1} = (V_{n+1}, E_{0,n+1})$.

Proof: We consider the points of $V_n$and V_{n+1}. It can be easily seen that there is a one-one correspondence between the points of the two sets V_n and V_{n+1} set by the rule $x_{i,j} \to x_{i,2j}$,if i is even, otherwise $x_{i,j} \to x_{i,2j-1}$.Clearly if there is a line between x_{i_1,j_1} and x_{i_2,j_2} then there exists a line between their corresponding elements. Thus G_n can be embedded in G_{n+1}.

Theorem 3.3: There always exists a path among the points of the set $\{x_{2^k-1,i}$|k=0, 1,2,…,n-1$\}$for some suitable values of i.

Proof: The proof is based on induction on V_n.It is clear that there exists a path among the points of the set$V_1 = \{x_{0,0}, x_{1,1}\}$. Let there exists a path among the points $\{x_{2^k-1,i}$|k=0, 1, 2,…, n-1$\}$ of V_{n-1}.To prove that there exists a line between the point $x_{2^{n-1}-1,i}$ *and* $x_{2^n-1,j}$ for some i,j. By the construction of the setV_n, it is obvious that the points are in fact $x_{2^{n-1}-1,2^n-2}$ *and* $x_{2^n-1,2^n-1}$. Hence there must exist a line among these two points, which concludes the theorem. It can be easily seen that degree of each element of the set $V_{n,1}$except $x_{0,0}$has degree 2, but $x_{0,0}$being the first point has degree 1 with respect to the edge set $E_{0,n}$.

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Source of Support: None Declared
Conflict of Interest: None Declared