Original Article

Exponential-Bayesian estimation based on maximum ranked set sampling with unequal samples using asymmetric loss function

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Abstract In this paper, we provide Bayesian estimation of the parameter of the exponential distribution based on maximum ranked set sampling with unequal samples (MRSSU). Under this method, we use linex loss function as asymmetric loss function and squared error loss function as symmetric loss function (SEL) to derive Bayesian estimate of the parameter of exponential distribution. The conjugate and Jeffreys-non informative prior distributions are used to study the performance of the obtained estimates. The efficiency of these estimates are compared with estimates based on simple random sampling (SRS) and ranked set sampling (RSS). It is shown that suggested estimators are more efficient than the estimators from SRS, and it is a good competitor to RSS.
Keywords: Bayes estimation, Conjugate prior, Jeffreys prior, Maximum ranked set sampling, Ranked set sampling.

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INTRODUCTION

The concept of ranked set sampling was first proposed by McIntyre[9] for situations where measuring the sample observations is not easy or it is costly and time consuming, but the ranking of items according to the variable of interest is relatively easy without considering actual measurement. In RSS; one first draws m^2 units at random from the population and partitions them in to m-sets of m-units. The m units in each set are ranked without making actual measurements. The first set of m-units are ranked and the lowest is chosen for actual quantification. From the second set of m-units, the unit ranked second and the lowest is measured. This process is continued until the unit ranked largest is measured from the m-th set. If a larger sample size is required then the procedure can be repeated for r times to obtain a sample of size $n = rm$. These chosen elements are called ranked set sample. Takahasi and Wakimoto[17] and Dell and Clutter[7] established statistical foundation for the theory of RSS. For more research work on parametric methods for RSS, see, for example, Stokes[16], Samawi et al.[13], Shaibu and Muttlak[15], Sengupta and Sujay Mukhuti[14]. Al-Omari et al.[2]. Some studies have investigated ranked set sampling (RSS) from Bayesian point of view. Al-Saleh and Muttlak[3] used RSS in Bayesian estimation for exponential and normal distributions to reduce Bayes risk. Zellner[19] studied Bayesian estimation

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for scalar least regression coefficient estimator using asymmetric loss functions. Al-Hadhrami and Al-Omari[1] discussed the Bayesian inference on the variance for normal distribution using moving extremes RSS. Sadek, Sultan and Balakrishnan[12], and Sadek and Alharbi[11] have obtained the Bayesian estimate of the exponential and Weibull distributions using SEL and LINEX loss functions respectively. They showed that the Bayesian estimators based on RSS are less biased and more efficient than the corresponding Bayesian estimators of SRS. Ghafoori et al.[8] studied the Bayesian two-sample prediction with progressively Type-II censored data for some life time models. Mohie El-Din, Kotb and Newer[10] studied the Bayesian estimation and prediction for Pareto distribution based on RSS. In this paper, we consider MRSSU method proposed by Biradar and Santosha[5] for Bayesian estimation of exponential distribution. In Section 2, some basic concepts are discussed. The Bayesian estimators under SEL and LINEX loss functions of the parameter of exponential distribution using SRS and MRSSU are presented in Section 3 and 4 respectively. Simulation results are presented in section 5.

BASIC CONCEPTS

Let $X_1, X_2, \ldots, X_m$, be sequence of independent and identically distributed (iid) random variables from exponential distribution with parameter θ, has a probability density function

$$f(x,\theta) = \theta e^{-\theta x}, x > 0, \theta > 0. \tag{1}$$

and its cumulative distribution function (cdf) is

$$F(x,\theta) = 1 - e^{-\theta x}, \quad x > 0, \theta > 0. \tag{2}$$

In order to derive the Bayesian estimators, the conjugate prior for θ is considered, i.e., $\theta : Gamma(\alpha, \beta)$ whose pdf is given by

$$\pi(\theta) = \frac{e^{-\theta\beta}\theta^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}}, \quad 0 < \theta < \infty, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are the hyper parameters. If $\alpha = \beta = 0$, then $\pi(\theta)$ becomes the Jeffreys prior. In Bayesian approach most of the studies use the squared error loss function (SEL) (see, Box and Tiao[6] and Berger[4]), as the basis of measuring estimators performance. Linear exponential (LINEX) loss function, introduced by Varian[18], is a asymmetric loss function and it is natural extension of SEL. The LINEX loss function for the parameter θ can be expressed as $L(\delta, c) = d(e^{c\delta} - c\delta - 1)$, where $\delta = (\hat{\theta} - \theta)$; $\hat{\theta}$ is an estimate of θ and $c \neq 0$, c and d are shape and scale parameters. The Bayes estimator of θ under the LINEX loss function, denoted by $\hat{\theta}_{Lnx}$, is the value which minimizes $E[L(\hat{\theta} - \theta)]$, and it is given by

$$\hat{\theta}_{Lnx} = -\frac{1}{c} lnE(e^{-c\theta}).$$

BAYES ESTIMATE BASED ON SRS

In this section, we derive the Bayesian estimate of the exponential parameter θ based on SRS. We use both conjugate and the non-informative prior and also we consider SEL and LINEX functions to derive the estimates. Let $p(\theta \mid \underline{x})$ denote the posterior density of θ, given SRS $\underline{X}$ is

$$p(\theta \mid \underline{x}) = \frac{\pi(\theta)\prod_{i=1}^{m} f(x_i \mid \theta)}{\int_0^{\infty} \pi(\theta)\prod_{i=1}^{m} f(x_i \mid \theta) d\theta}$$

$$= \frac{e^{-\theta\beta}\theta^{\alpha-1}\prod_{i=1}^{m}\theta e^{-\theta x_i}}{\int_0^\infty e^{-\theta\beta}\theta^{\alpha-1}\prod_{i=1}^{m}\theta e^{-\theta x_i}d\theta}$$

then,

$$= \frac{\theta^{m+\alpha-1}e^{-\theta(\sum_{i=1}^{m}x_i+\beta)}}{\Gamma(m+\alpha)(\sum_{i=1}^{m}x_i+\beta)^{-(m+\alpha)}} \quad . \qquad (4)$$

The Bayesian estimate of θ based on SEL function is

$$\theta^*_{Sel} = E(\theta \mid X) = \int_0^\infty \theta p(\theta \mid \underline{x})d\theta$$

$$= \frac{\int_0^\infty \theta^{(m+\alpha+1)-1}e^{-\theta(\sum_{i=1}^{m}x_i+\beta)}}{\Gamma(m+\alpha)(\sum_{i=1}^{m}x_i+\beta)^{-(m+\alpha)}}d\theta$$

Therefore,

$$= \frac{m+\alpha}{\sum_{i=1}^{m}x_i+\beta}. \qquad (5)$$

The Bayesian estimate of θ based on LINEX loss function is

$$\theta^*_{Lnx} = -\frac{1}{c}lnE(e^{-c\theta})$$

$$= -\frac{1}{c}ln\left[\frac{\int_0^\infty \theta^{m+\alpha-1}e^{-\theta(\sum_{i=1}^{m}x_i+\beta+c)}d\theta}{\Gamma(m+\alpha)(\sum_{i=1}^{m}x_i+\beta)^{-(m+\alpha)}}\right]$$

then,

$$= \left(\frac{m+\alpha}{c}\right)ln\left(\frac{\sum_{i=1}^{m}x_i+\beta}{\sum_{i=1}^{m}x_i+\beta+c}\right) \quad . \qquad (6)$$

The non-informative prior distribution of the parameter θ obtained from (3) which is given by $\pi(\theta) \propto \frac{1}{\theta},\ \theta > 0$. Then, the posterior density can be expressed as

$$p(\theta \mid \underline{x}) = \frac{\frac{1}{\theta}\prod_{i=1}^{m} f(x_i \mid \theta)}{\int_0^\infty \frac{1}{\theta}\prod_{i=1}^{m} f(x_i \mid \theta) d\theta}$$

$$= \frac{\theta^{m-1} e^{-\theta\sum_{i=1}^{m} x_i}}{\int_0^\infty \theta^{m-1} e^{-\theta\sum_{i=1}^{m} x_i} d\theta}$$

then,

$$= \frac{\theta^{m-1} e^{-\theta\sum_{i=1}^{m} x_i} (\sum_{i=1}^{m} x_i)^m}{\Gamma(m)}.$$

We can observe that the posterior distribution of θ is a gamma distribution with parameter $(m, \sum_{i=1}^{m} x_i)$. The Bayesian estimate of θ based on squared error loss function is:

$$\hat{\theta}_{Sel}^{j} = E(\theta \mid \underline{X})$$

$$= \frac{(\sum_{i=1}^{m} x_i)^m \int_0^\infty \theta^{(m+1)-1} e^{-\theta\sum_{i=1}^{m} x_i} d\theta}{\Gamma(m)}$$

$$= \frac{\Gamma(m+1)}{\Gamma(m)(\sum_{i=1}^{m} x_i)}$$

$$= \frac{m}{\sum_{i=1}^{m} x_i} \quad . \tag{7}$$

The Bayesian estimation of θ based on LINEX loss function is

$$\hat{\theta}_{Lnx}^{j} = -\frac{1}{c} ln[E(e^{-c\theta})]$$

$$= -\frac{1}{c} ln\left[\frac{(\sum_{i=1}^{m} x_i)^m}{\Gamma(m)} \int_0^\infty \theta^{m-1} e^{-\theta(\sum_{i=1}^{m} x_i + c)} d\theta\right]$$

therefore,

$$=\left(\frac{m}{c}\right)ln\left[\frac{\sum_{i=1}^{m}x_i}{\sum_{i=1}^{m}x_i+c}\right] \quad . \tag{8}$$

BAYES ESTIMATE BASED ON MRSSU

In this section, we derive the Bayesian estimate of θ based on MRSSU using squared error and linex loss functions. Let $\{X_{i1},X_{i2},\ldots,X_{ii}\}$, $i=1,\ldots,m$ be m sets of random samples from X, and they are independent. Denote $Y_i = Max\{X_{i1},X_{i2},\ldots,X_{ii}\}$, $i=1,\ldots,m$. Then $\{Y_1,Y_2,\ldots,Y_m\}$ be a MRSSU of X. Note that the elements of this samples are independent. It is assumed through out this paper that the judgemental ranking is perfect, i.e., the visual ranking is the same as the actual ranking. Then, the density of Y_i has the same density as the i-th order statistic (maximum) of an SRS of size i from $f(x,\theta)$, i.e., Y_i has the density

$$f_i(y\mid\theta) = i[F(y,\theta)]^{i-1} f(y,\theta).$$

Let MRSSU be drawn from an exponential distribution defined in (1), then the density function of Y_i is

$$f(y_i\mid\theta) = i[1-e^{-\theta \quad y_i}]^{i-1}\theta e^{-\theta y_i}$$

$$=\sum_{q=0}^{i-1} i\binom{i-1}{q}(-1)^q\theta e^{-\theta y_i(q+1)}, \quad y_i>0,\theta>0.$$

Then, the joint density of MRSSU due to independence of $y_{i'}$ s is given by

$$f(\underline{y}\mid\theta) = \prod_{i=1}^{m} f(y_i\mid\theta) = \prod_{i=1}^{m}\sum_{q=0}^{i-1} i\binom{i-1}{q}(-1)^q\theta e^{-\theta y_i(q+1)}$$

$$=\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left[\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right]\theta^m e^{-\theta\sum_{i=1}^{m}y_i(k_i+1)} .$$

CONJUGATE PRIOR FOR θ

Let $\Pi(\theta\mid\underline{y})$ denote the posterior density of θ, given MRSSU. Then, the posterior density of θ is

$$\Pi(\theta\mid\underline{y}) = \frac{\pi(\theta)f(\underline{y}\mid\theta)}{\int_0^\infty \pi(\theta)f(\underline{y}\mid\theta)d\theta}$$

$$=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})e^{-\theta(\sum_{i=1}^{m}y_i(k_i+1)+\beta)}\theta^{m+\alpha-1}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\int_0^\infty e^{-\theta(\sum_{i=1}^{m}y_i(k_i+1)+\beta)}\theta^{m+\alpha-1}d\theta}$$

Therefore,

$$=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\theta^{m+\alpha-1}e^{-\theta\left(\sum_{i=1}^{m} y_i(k_i+1)+\beta\right)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\Gamma(m+\alpha)\left[\sum_{i=1}^{m} y_i(k_i+1)+\beta\right]^{-(m+\alpha)}} \quad . \tag{9}$$

The Bayes estimate of θ under SEL function is .

$$\tilde{\theta}_{Sel} = E(\theta \mid \underline{Y}) = \int_0^\infty \theta\Pi(\theta \mid \underline{y})d\theta$$

$$=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\int_0^\infty \theta^{(m+\alpha+1)-1}e^{-\theta\left(\sum_{i=1}^{m} y_i(k_i+1)+\beta\right)}d\theta}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\Gamma(m+\alpha)\left[\sum_{i=1}^{m} y_i(k_i+1)+\beta\right]^{-(m+\alpha)}}$$

$$=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)(m+\alpha)\left(\sum_{i=1}^{m} y_i(k_i+1)+\beta\right)^{-(m+\alpha+1)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\left[\sum_{i=1}^{m} y_i(k_i+1)+\beta\right]^{-(m+\alpha)}} \quad . \tag{10}$$

To obtain the Bayesian estimate of θ based on LINEX loss function, we need to calculate the posterior expectation of $e^{-c\theta}$ from (9), we have

$$E(e^{-c\theta})=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\int_0^\infty \theta^{m+\alpha-1}e^{-\theta\left(\sum_{i=1}^{m} y_i(k_i+1)+\beta+c\right)}d\theta}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\Gamma(m+\alpha)\left[\sum_{i=1}^{m} y_i(k_i+1)+\beta\right]^{-(m+\alpha)}}$$

Therefore,

$$=\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)(m+\alpha)\left(\sum_{i=1}^{m} y_i(k_i+1)+\beta+c\right)^{-(m+\alpha+1)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}\left(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right)\left[\sum_{i=1}^{m} y_i(k_i+1)+\beta\right]^{-(m+\alpha)}}. \tag{11}$$

Now from (11), the Bayesian estimate of $\tilde{\theta}$ on LINEX is

$$\tilde{\theta}_{Lnx}(\underline{y}) = -\frac{1}{c}ln[E(e^{-c\theta})] \quad . \tag{12}$$

NON-INFORMATIVE PRIOR FOR θ

The non-informative prior distribution of the parameter θ is given by $\pi(\theta) \propto \frac{1}{\theta}$, $\theta > 0$. The posterior density of θ for MRSSU can be derived as

$$\Pi(\theta \mid \underline{y}) = \frac{\frac{1}{\theta} f(\underline{y} \mid \theta)}{\int_0^\infty \frac{1}{\theta} f(\underline{y} \mid \theta) d\theta}$$

$$= \frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})e^{-\theta(\sum_{i=1}^{m} y_i(k_i+1)+\beta)}\theta^{m+\alpha-1}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\int_0^\infty e^{-\theta(\sum_{i=1}^{m} y_i(k_i+1)+\beta)}\theta^{m+\alpha-1}d\theta}$$

$$= \frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\theta^{m-1}e^{-\theta\sum_{i=1}^{m} y_i(k_i+1)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\Gamma m[\sum_{i=1}^{m} y_i(k_i+1)]^{-m}}. \quad (13)$$

From (13), the Bayes estimate of θ under SEL function is

$$\tilde{\theta}_{Sel}^{j} = E(\theta \mid \underline{Y}) = \int_0^\infty \theta \Pi(\theta \mid \underline{y}) d\theta$$

$$= \frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\int_0^\infty \theta^{(m+1)-1}e^{-\theta\sum_{i=1}^{m} y_i(k_i+1)}d\theta}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\Gamma m[\sum_{i=1}^{m} y_i(k_i+1)]^{-m}}$$

$$= \frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})m(\sum_{i=1}^{m} y_i(k_i+1))^{-(m+1)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})[\sum_{i=1}^{m} y_i(k_i+1)]^{-m}}. \quad (14)$$

To obtain the Bayesian estimate of θ based on LINEX loss function, we need to calculate the posterior expectation of $e^{-c\theta}$ from (13), we have

$$E(e^{-c\theta}) = \frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\int_0^\infty \theta^{m-1}e^{-\theta(\sum_{i=1}^{m} y_i(k_i+1)+c)}d\theta}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})\Gamma m[\sum_{i=1}^{m} y_i(k_i+1)]^{m}}.$$

Therefore,

$$=-\frac{1}{c}ln\left[\frac{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})m(\sum_{i=1}^{m} y_i(k_i+1)+c)^{-(m+1)}}{\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_m=0}^{m-1}(\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i})[\sum_{i=1}^{m} y_i(k_i+1)]^{m}}\right]. \quad (15)$$

Now from (15), the Bayesian estimate of θ on LINEX is

$$\tilde{\theta}^{j}_{Lnx}(\underline{y}) = -\frac{1}{c}ln[E(e^{-c\theta})].$$

BAYES ESTIMATE BASED ON r - CYCLE MRSSU

Let $Y_{1:1}, Y_{1:2}, \ldots, Y_{2:m}; Y_{2:1}, Y_{2:2}, \ldots, Y_{2:m}; \ldots; Y_{r:1}, Y_{r:2}, \ldots, Y_{r:m};$ be the r cycle MRSSU from the exponential distribution and the prior pdf, $Gamma:(\alpha,\beta)$.

Then the joint density is given by

$$g(\underline{y}|\theta)=\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\left[\prod_{i=1}^{m} i\binom{i-1}{k_i}(-1)^{k_i}\right]\theta^m e^{-\theta\sum_{i=1}^{m} y_i(k_i+1)}, \quad y_i>0$$

$$=\left[\sum_{k_1^1=0}^{0}\sum_{k_2^1=0}^{1}\cdots\sum_{k_m^1=0}^{m-1}\right]\left[\sum_{k_1^2=0}^{0}\sum_{k_2^2=0}^{1}\cdots\sum_{k_m^2=0}^{m-1}\right]\cdots\left[\sum_{k_1^r=0}^{0}\sum_{k_2^r=0}^{1}\cdots\sum_{k_m^r=0}^{m-1}\right]$$

$$\left[\prod_{l=1}^{r}\prod_{i=1}^{m} i\binom{i-1}{k_i^l}(-1)^{k_i^l}\right]\theta^{rm} e^{-\theta\sum_{l=1}^{r}\sum_{i=1}^{m} y_{l_i}(k_i^l+1)}$$

$$=\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\theta^{rm} e^{-\theta Q_{k_i^l}}, \quad y_{l_i}>0$$

where $\xi_{k_i^l}=\left[\prod_{l=1}^{r}\prod_{i=1}^{m} i\binom{i-1}{k_i^l}(-1)^{k_i^l}\right]$ and $Q_{k_i^l}=\sum_{l=1}^{r}\sum_{i=1}^{m} y_{l_i}(k_i^l+1)$.

The posterior density of θ for MRSSU can be derived as

$$\Pi(\theta|\underline{y})=\frac{\pi(\theta)f(\underline{y}|\theta)}{\int_0^\infty \pi(\theta)f(\underline{y}|\theta)d\theta}$$

$$=\frac{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\theta^{rm+\alpha-1}e^{-\theta(Q_{k_i^l}+\beta)}}{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\int_0^\infty\theta^{rm+\alpha-1}e^{-\theta(Q_{k_i^l}+\beta)}d\theta}$$

$$=\frac{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\theta^{rm+\alpha-1}e^{-\theta(Q_{k_i^l}+\beta)}}{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\Gamma(rm+\alpha)(Q_{k_i^l}+\beta)^{-(rm+\alpha)}}. \tag{16}$$

From (16), the Bayesian estimate of θ based on SEL function is obtained as

$$\tilde{\theta}_{Sel}(\underline{Y}^{(r)})=E(\theta\,|\,\underline{Y})$$

$$=\frac{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\int_0^\infty\theta^{(rm+\alpha+1)-1}e^{-\theta(Q_{k_i^l}+\beta)}d\theta}{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}\Gamma(rm+\alpha)(Q_{k_i^l}+\beta)^{-(rm+\alpha)}}$$

$$=\frac{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}(rm+\alpha)(Q_{k_i^l}+\beta)^{-(rm+\alpha+1)}}{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}(Q_{k_i^l}+\beta)^{-(rm+\alpha)}}.$$

Using the non-informative Jefferys prior in (3), we obtain the Bayesian estimate based on the SEL function is

$$\tilde{\theta}_{Sel}^{j}(\underline{Y}^{(r)})=\frac{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}(rm)[Q_{k_i^l}]^{-(rm+1)}}{\left[\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right]\xi_{k_i^l}[Q_{k_i^l}]^{-(rm)}}. \tag{17}$$

From (16), the Bayesian estimate of θ based on LINEX function is

$$\tilde{\theta}_{Lnx}(\underline{Y})=-\frac{1}{c}ln\left[\frac{\int_0^\infty e^{-c\theta}\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}\theta^{rm+\alpha-1}e^{-\theta(Q_{k_i^l}+\beta)}}{\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}\Gamma(rm+\alpha)(Q_{k_i^l}+\beta)^{-(rm+\alpha)}}\right]$$

$$=-\frac{1}{c}ln\left[\frac{\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}(rm+\alpha)(Q_{k_i^l}+\beta+c)^{-(rm+\alpha+1)}}{\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}(Q_{k_i^l}+\beta)^{-(rm+\alpha)}}\right]. \quad (18)$$

Using the non-informative Jefferys prior in (3), we obtain the Bayesian estimate based on the SEL function to be

$$\tilde{\theta}_{Lnx}^{j}(\underline{Y})=-\frac{1}{c}ln\left[\frac{\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}(rm)(Q_{k_i^l}+c)^{-(rm+1)}}{\left(\prod_{l=1}^{r}\sum_{k_1^l=0}^{0}\sum_{k_2^l=0}^{1}\cdots\sum_{k_m^l=0}^{m-1}\right)\xi_{k_i^l}(Q_{k_i^l})^{-(rm)}}\right]. \quad (19)$$

SIMULATION RESULTS

In order to evaluate the behaviour of the Bayesian estimates based on SRS, RSS and MRSSU, we perform a simulation using R-software version 3.1.1 for different values of m =3, 4 and 5 according to the following steps:

1. Generate SRS and RSS and MRSSU samples of size m from the exponential distribution with $\theta=2$ for $r=1$ cycle.
2. Compute the Bayesian estimates derived in Section 3 and 4 for SRS and MRSSU, respectively and for RSS [See, equations of Sadek, Sultan and Balakrishnan[12]].
3. Repeat the above Steps 1 and 2 for 1000 runs.
4. Compute, $Bias(\hat{\theta},\theta)=\bar{\hat{\theta}}-\theta$ and MSE of all the etimates, where $\bar{\hat{\theta}}$ is the average of the 1000 estimates of θ and θ is the value that is used in the simulation.
5. Let e_1 and e_2 represents relative efficiency of Bayesian estimate of θ based on MRSSU with respect to SRS, for SEL and Linex functions respectively be defined by

$$e_1=\frac{MSE_{SRS}(\theta_{Sel})}{MSE_{MRSSU}(\theta_{Sel})} \quad and \quad e_2=\frac{MSE_{SRS}(\theta_{Lnx})}{MSE_{MRSSU}(\theta_{Lnx})}.$$

Similarly, e_3 and e_4 represents relative efficiency of Bayesian estimate of θ based on MRSSU with respect to RSS, for SEL and Linex functions respectively is defined by

$$e_3=\frac{MSE_{RSS}(\theta_{Sel})}{MSE_{MRSSU}(\theta_{Sel})} \quad and \quad e_4=\frac{MSE_{RSS}(\theta_{Lnx})}{MSE_{MRSSU}(\theta_{Lnx})}.$$

Bias, MSE and Relative efficiency of the Bayesian estimates based on SRS, RSS and MRSSU, when $\theta=2$, $\alpha=1$, $\beta=1$ of size m =3, 4 and 5 are represented in Table 1-3.

From Table 1, it is observed that the Bayesian estimates of θ are all biased. Next, we observed that the Bayesian estimates based on Jeffreys prior are less biased than gamma prior. Also, we observe that the Bayesian estimates based on RSS and MRSSU are less biased than the corresponding SRS Bayesian estimates. From Table 2, it is observed that the mean squared error of all estimates decreases when sample size increases. Next, we observe that the Bayesian estimates based on RSS and MRSSU have a much smaller mean squared error than the corresponding Bayesian estimates based on SRS in all cases considered and MRSSU MSE is lesser than RSS MSE for m=5. From Table 3, we observed that the computed relative efficiency (e_1,e_2) of MRSSU Bayesian estimator w.r.t. SRS Bayesian estimators are greater than 1 and increases with m. Therefore the Bayesian estimator based on MRSSU is more efficient than the corresponding SRS Bayesian estimator for Jeffrey and gamma prior. Also, it is observed that the relative efficiencies (e_3, e_4) of MRSSU Bayesian estimator w.r.t. RSS decreases in the case of 3≤m≤4 as the sample size m increases. We notice that the relative efficiency (e_3, e_4) is greater than 1 for m=5, in this case estimator using MRSSU is better than RSS. This is because RSS of the same size m uses more ranked units than MRSSU of the same size, for example RSS of size 5 uses 25 ranked units, whereas MRSSU of the same size uses only 15 ranked units. Therefore we conclude that for large sample size m, MRSSU Bayesian estimators are good competitor to RSS Bayesian estimator.

Table 1: Bias of the Bayesian estimates, when θ=2, α =1, β =1 and c=1, -1.

m	Bias(θ_{Sel})			Bias(θ_{Sel})			c	Bias(θ_{Lnx})			Bias(θ_{Lnx})		
	Jeffrey prior			Gamma prior				Jeffrey prior			Gamma prior		
	SRS	RSS	MRSSU	SRS	RSS	MRSSU		SRS	RSS	MRSSU	SRS	RSS	MRSSU
3	0.2218	0.0934	0.0937	0.2151	0.1204	0.1205	1	0.1150	0.0573	0.0566	0.1450	0.0880	0.0874
							-1	0.3542	0.1406	0.1429	0.3200	0.1597	0.1609
4	0.1510	0.0468	0.0576	0.1696	0.0691	0.0803	1	0.0953	0.0288	0.0379	0.1278	0.0517	0.0613
							-1	0.2565	0.0669	0.0800	0.2441	0.0884	0.1015
5	0.1196	0.0316	0.0363	0.1404	0.0478	0.0538	1	0.0600	0.0202	0.0238	0.0924	0.0366	0.0415
							-1	0.1568	0.0438	0.0498	0.1720	0.0598	0.0671

Table 2: MSE of the Bayesian estimates, when θ=2, α =1, β =1 and c=1, -1.

M	MSE(θ_{Sel})			MSE(θ_{Sel})			c	MSE(θ_{Lnx})			MSE(θ_{Lnx})		
	Jeffrey prior			Gamma prior				Jeffrey prior			Gamma prior		
	SRS	RSS	MRSSU	SRS	RSS	MRSSU		SRS	RSS	MRSSU	SRS	RSS	MRSSU
3	0.4269	0.0916	0.0960	0.1957	0.0797	0.0824	1	0.1828	0.0650	0.0678	0.1143	0.0596	0.0614
							-1	1.0034	0.1467	0.1587	0.4040	0.1121	0.1167
4	0.2138	0.0379	0.0429	0.1382	0.0380	0.0430	1	0.1151	0.0317	0.0355	0.0930	0.0318	0.0356
							-1	0.4381	0.0464	0.0533	0.2253	0.0461	0.0528
5	0.1696	0.0264	0.0249	0.1145	0.0266	0.0256	1	0.0773	0.0235	0.0219	0.0669	0.0236	0.0224
							-1	0.2051	0.0301	0.0287	0.1357	0.0304	0.0296

Table 3: Relative efficiency when θ=2, α =1, β =1 and c=1, -1.

M	eff-Jeffrey (Sel)		eff-Gamma (Sel)		c	eff-Jeffrey (Lnx)		eff-Jeffrey (Lnx)	
	e_1	e_3	e_1	e_3		e_2	e_4	e_2	e_4
3	4.4469	0.9542	2.3750	0.9672	1	2.6962	0.9587	1.8616	0.9707
					-1	6.3226	0.9244	3.4619	0.9606
4	4.9837	0.8834	3.2140	0.8837	1	3.2423	0.8930	2.6124	0.8933
					-1	8.2195	0.8705	4.2670	0.8731
5	6.8112	1.0602	4.4727	1.0391	1	3.5297	1.0731	2.9866	1.0536
					-1	7.1463	1.0488	4.5845	1.0270

REFERENCES

1. Al-Hadhrami, S.A. and Al-Omari, A.I. “Bayesian Inference on the Variance of Normal Distribution Using Moving Extremes Ranked Set Sampling”, J. of Mod. Appl. Stat. Math., 8(1), 273-281, 2009.
2. Al-Omari, A.I. Jaber, K. and Al-Omari, A. “ Modified Ratio-Type Estimators of the Mean Using Extreme Ranked Set Sampling”, J of Math. and Stat, 4(3), 150-155, 2008.
3. Al-Saleh, M.F. and Muttlak, H.A., “A Note in Bayesian Estimation Using Ranked Set Sampling”, Pakistan Journal of Statistics, 14, 49-56, 1998.
4. Berger, J.O., “Statistical Decision Theory and Bayesian Analysis”, Springer-Verlag, New Yark, 1985.
5. Biradar, B.S. and Sanotsha, C.D., “Estimation of the Mean of the Exponential Distribution Using Maximum Ranked Set Sampling with Unequal Samples”, Open journal of Statistics, Scientific Research, 4, 2014.
6. Box, G.E.P. and Tiao, G.C., “ Bayesian Inference in Statistical Analysis”, Addison Wesly, 1973.

7. Dell, T.R. and Clutter, J.L., " Ranked Set Sampling Theory with Order Statistics Background", Biometrics, 28, 545-555, 1972.
8. Ghafoori, S., Habibi Rad, A. and Doostparast, M., "Bayesian Two-Sample Prediction with Type-II Censored Data for Some Lifetime Models", JIRSS, 10(1), 63-86, 2011.
9. McIntyre, G.A., " A Method for Unbiased Selective Sampling Using Ranked Sets", Australian Journal of Agricultural Research, 3, 385-390, 1952.
10. Mohie El-Din, M.M., Kotb, M.S. and Newer, H.A., "Bayesian Estimation and Prediction for Pareto Distribution Based on Ranked Set Sampling", Journal of Statistics Applcations and Probability, 4, No.2, 211-221, 2015.
11. Sadek, A. and Alharbi, F. , "Weibull-Bayesian Analysis Based on Ranked Set Sampling", Economic Quality Control, 1-15, 2010.
12. Sadek, A., Sultan, K.S. and Balakrishnan, N., "Bayesian Estimation Based on Ranked Set Sampling Using Asymmetric Loss Function", Bull. Malays. Math. Sci. Soc., 2009.
13. Samwi, H., Ahmad, M. and Abu-Dayyeh, W. "Estimating the Population Mean Using Extreme Ranked Set Sampling", Biom. Journal., 38, 577-586, 1996.
14. Sengupta, S. and Sujay Mukhuti, "Unbiased Variance Estimation in a Simple Exponential Population Using Ranked Set Samples", Journal of Statistical and Inference, 136, 1526-1553, 2006.
15. Shaibu, A.B. and Muttlak, H.A., " Estimating the Parameters of the Normal, Exponential and Gamma Distributions Using Median and Extreme Ranked Set Samples", Statistics., 1, 75-98, 2004.
16. Stokes, L., "Parametric Ranked Set Sampling", Ann. Inst. Statist. Math., 47, 465-482, 1995.
17. Takahasi, K. and Wakimoto, K., "On Unbiased Estimates of the Population Mean Based on the Sample Stratified by Means of Ordering", Annals of the Institute of Statistical Mathematics, 20, 1-31, 1968.
18. Varian, H.R., "A Bayesian Approach to Real Estate Assessment", North Holland, Amsterdam, 195-208, 1975.
19. Zellner, A., "Bayesian Estimation and Prediction Using asymmetric loss Functions", J. Amer. Statist. Assoc., 81, 446-451, 1986.

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