Original Article

Analysis of various fractal dimensions at bifurcation points in a one dimensional non linear map

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Abstract In this paper a unimodal map exhibiting period doubling bifurcation scenario has been considered. First of all period doubling bifurcation points have been calculated numerically and hence the accumulation point has been obtained. Secondly, box counting dimension, information dimension and correlation dimensions have been calculated on the set of attractors at the bifurcation parameters and have been found to be fractional.

Key words: Unimodal map/Bifurcation/Accumulation point/Fractal Dimension
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INTRODUCTION

Discovery of Feigenbaum Universality in case of one dimensional unimodal maps confirmed the onset of chaos[4,5]**,** however the detection of the accumulation point is generally confirmed by calculating Lyapunov exponents, fractal dimensions e.t.c[7,9] Some aspects of chaos is the occurrence of fractional dimensions in the set of strange attractor which is accompanied by sensitivity to initial condition[1]. Sensitivity to initial condition is often detected by positive Lyapunov exponents[9]. Various types fractal dimensions viz. capacity dimension, correlation dimension, information dimensioncan be calculated to detect the fractal property of the attractor set and for this an easy way is to calculate the generalized correlation dimension[6,7].

In this paper, a difference equation $x_{n+1} = f(x_n)$ where $f(x) = ax^2 - x^3$, a being a control parameter is considered. In the first section an effort has been given to calculate the bifurcation points with the help of some numerical technique and with the help of these bifurcation points the onset of chaos i.e. the accumulation point has been calculated[7].

In the second section various fractal dimensions viz capacity,…. have been calculated at each of the bifurcation points as well as at the accumulation point[6].

DYNAMIC SCENARIO OF THE MODEL

The discrete model for studying the dynamic scenario has been considered as:

$x_{n+1} = f(x_n)$ where $f(x) = ax^2 - x^3$ ……………………………….(2.1)

The map is unimodal having the critical point as $\frac{2a}{3}$. The domain and range of the map has been considered as $[0, a]$ and $[0, \frac{4a^3}{27}]$ respectively. Fixed points of f are $x = 0$, $x = \frac{1}{2}\left(a - \sqrt{a^2-4}\right)$, $x = \frac{1}{2}\left(a + \sqrt{a^2-4}\right)$. The fixed point

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$\frac{1}{2}\left(a-\sqrt{a^2-4}\right)$ is stable for $-\frac{4}{\sqrt{3}}<a<-2$ and the fixed point $\frac{1}{2}\left(a+\sqrt{a^2-4}\right)$is stable for $2<a<\frac{4}{\sqrt{3}}$. So when $a>\frac{4}{\sqrt{3}}$ the fixed point $\frac{1}{2}\left(a+\sqrt{a^2-4}\right)$ becomes unstable and birth of periodic points of period 2 occurs**[7].**

The governing equations for obtaining the nth bifurcation point in the period doubling scenario are :

$f^n(x)=x$ ....(2.2)

$\frac{d^n}{dx^n}f(x)=-1$ ....(2.3)

The bifurcation diagram obtained as

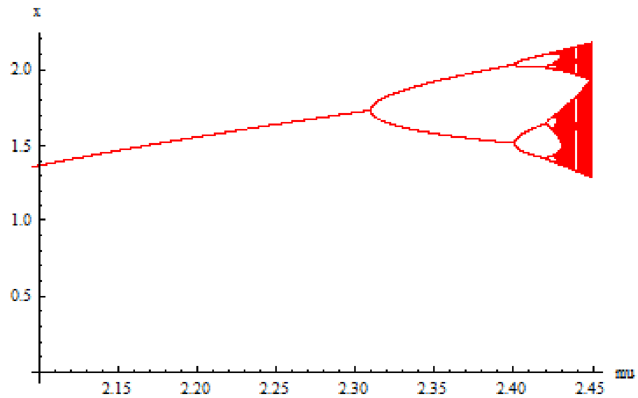

Fig 2.4: Showing the period doubling bifurcation

With the help of bisection method or Newton Raphson method the bifurcation points are calculated .The following table gives the bifurcation values and Feigenbaum delta at those points[4,5].

Table 2.5:Showing values of bifurcation point for different periods and corresponding Feigenbaum delta value

Sl No	PERIOD	BIFURCATION PARAMETER	Feigenbaum delta
1	1	2.30940107076850	
2	2	2.40075979942170	
3	4	2.42095534000697	
4	16	2.42530609392309	4.523707710494955
5	32	2.42530609392301	4.64185
6	64	2.42643893807937	4.66331
7	128	2.42648174618237	4.66796
8	256	2.42649091447946	4.66893
9	512	2.42649287805310	4.66914
10	1024	2.42649329859068	4.66919
11	2048	2.42649338865697	4.6692
12	4096	2.42649340794640	4.6692
13	2^{13}	2.42649341207761	4.6692
14	2^{14}	2.42649341296240	4.6692
15	2^{15}	2.42649341315189	4.66914
16	2^{16}	2.42649341319246	4.66933
17	2^{17}	2.42649341320115	4.67066
18	2^{18}	2.42649341320310	4.66864
19	2^{19}	2.42649341320344	4.45639

The accumulation point is obtained as 2.426493413203533 using the formula $A_\infty=\frac{A_2-A_1}{\partial}+A_2$ **[7].**

CALCULATION OF DIMENSIONS OF THE MODEL

To measure the dimension of the attractor set at various bifurcation parameters we study information dimension, box counting dimension and correlation dimension. For easier calculation, we take the help of generalized correlation sum G_q (N, R) **[2,6]**.

$$G_q(N,R)=\left[\frac{1}{N}\sum_{j=1}^{N}\left[\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}\theta\left(R-\left|x_j-x_k\right|\right)\right]^{q-1}\right]^{\frac{1}{q-1}} \text{.........(3.1)}$$

Where $\theta\left(R - |x_j - x_k|\right) = 1$ if $|x_j - x_k| < R$
$= 0$ if $|x_j - x_k| > R$,

Here, x_j represents the points in attractor set, N is the number of points from the attractor set to be considered for computation.

Further if $\lim_{N\to\infty} G_q(N,R) = G_q(R)$, then

$$D_q = \lim_{R\to 0} \frac{\log G_q(R)}{\log R} \quad \ldots\ldots(3.2)$$

We can see that D_q can be calculated by calculating the generalized correlation sum for any value of q. In particular for q= 0, 1, 2, from equation (3.2) we can get box counting, information and correlation dimensions respectively[2,6].

Table 3.3: Showing correlation, capacity counting and information dimension values at different bifurcation point as well as the accumulation point

Bifurcation Point	Correlation dimension	Capacity dimension	Information dimension
1st	0.886062±0.0560411	0.619973±0.0310977	0.871432±0.0729271
2nd	0.874087±0.0601341	0.609418+0.0260449	0.87494±0.0278821
3rd	0.846431±0.0427153	0.601587±0.00911772	0.844317±0.0277732
4th	0.766005±0.0484502	0.588318±0.0223249	0.775496±0.0507602
5th	0.541872±0.0889689	0.618134±0.0582998	0.690836±0.0671456
6th	0.530912±0.0822595	0.618355±0.0597909	0.637085±0.0487259
7th	0.459791±0.064768	0.79328±0.0754558	0.559285±0.0687621
Accumulation point	0.499406±0.0621438	0.562222±0.0881152	0.528257±0.0550878

Slope of the following diagram represents the dimensional value.

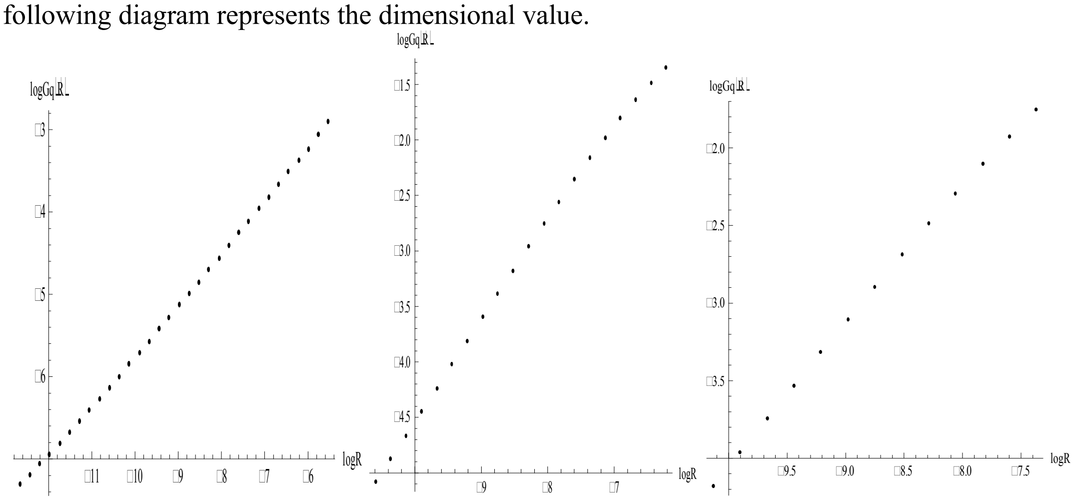

Fig 3.4 : $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the first bifurcation point

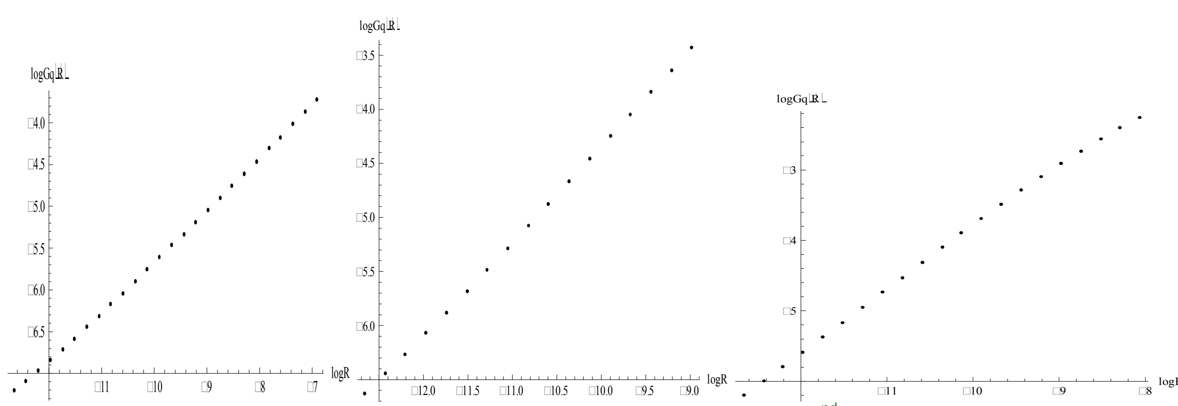

Fig3.5: $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 2nd bifurcation point

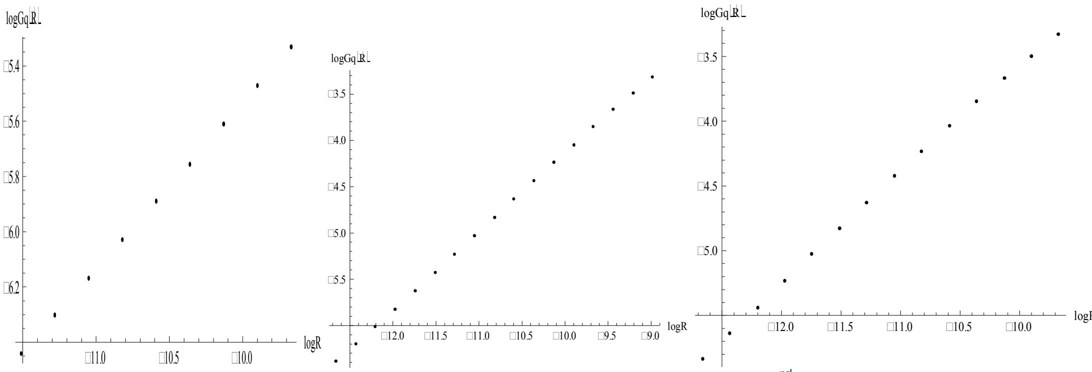

Fig3.6 : $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 3rd bifurcation point

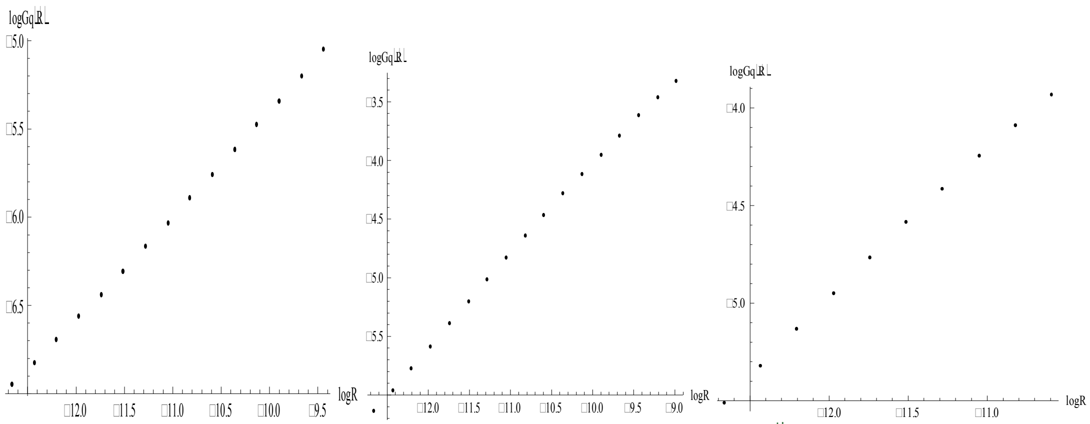

Fig3.7 : $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 4th bifurcation point

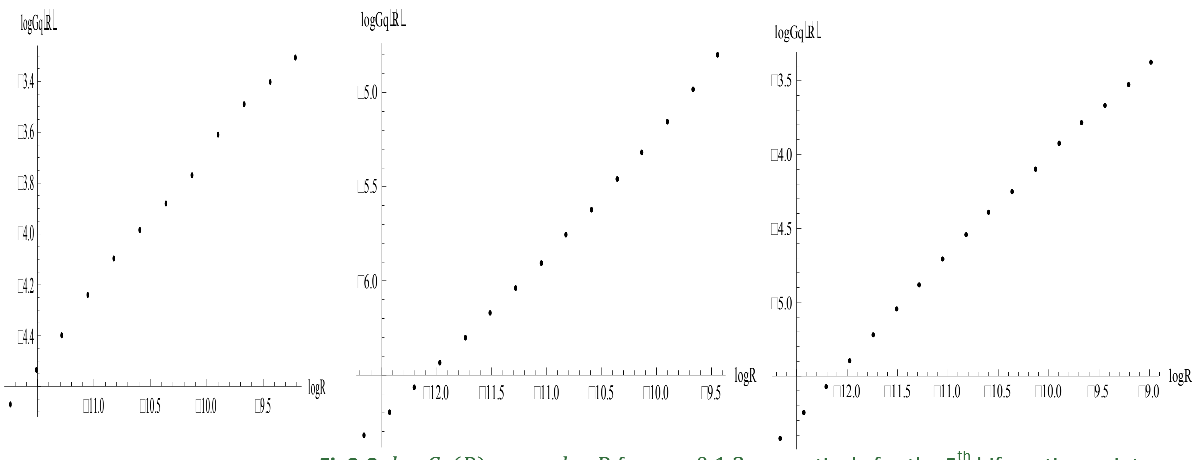

Fig3.8 :$logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 5th bifurcation point

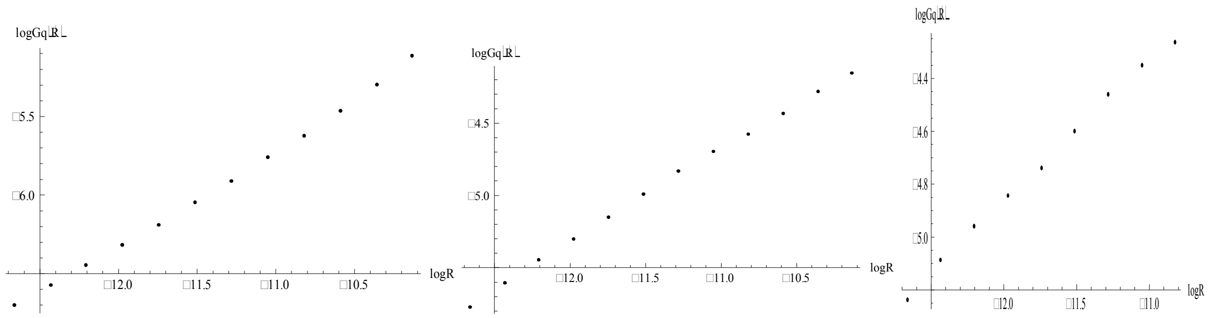

Fig3.9 : $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 6th bifurcation point

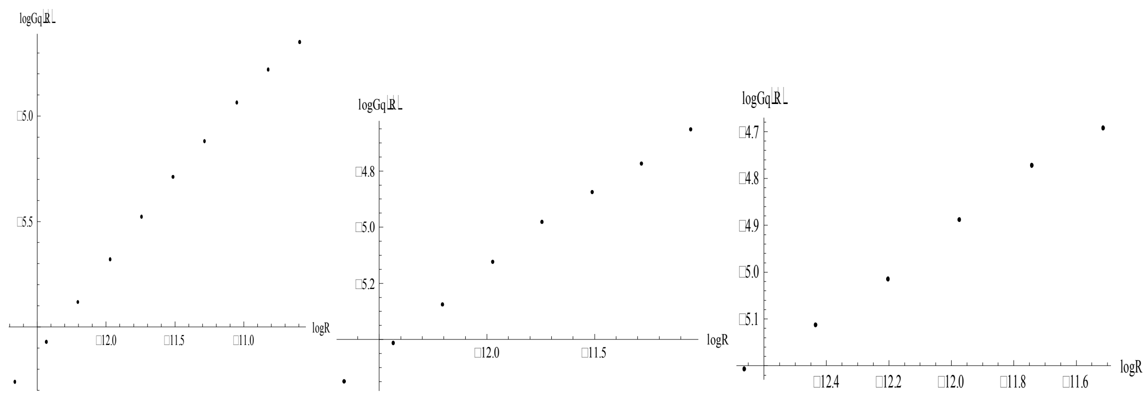

Fig3.10: $logG_q(R)$ versus $logR$ for $q = 0,1,2$ respectively for the 7th bifurcation point

CONCLUSION

It has been observed that the dimension values are fractional at the bifurcation as well as accumulation point maintaining a good agreement with the literature[4,5,8].

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