Original Article

A generalisation of almost sure limit sets of vector sequences of moving maxima

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Abstract We obtain almost sure limit sets of vector sequences of moving maxima and moving second maxima of a sequence of independent negative binomial random variable when $p = \infty$.
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INTRODUCTION

Let $\{ X_n, \ n \geq 1 \}$ be sequence of independent negative binomial random variables with probability mass function (p.m.f.)

$$P(X=k) = \begin{bmatrix} k-1 \\ r-1 \end{bmatrix} a^r(1-a)^{k-r},$$

$k = r, r+1, \ldots\ldots, \quad 0 < a < 1$

Define moving maxima and moving second maxima by

$$Y^*_{k(n)} = \max \{ X_{n-k(n)+1}, X_{n-k(n)+2}, \ldots, X_n \}$$

$$S^*_{k(n)} = \text{second largest} \{ X_{n-k(n)+1}, X_{n-k(n)+2}, \ldots, X_n \}$$

where $k(n)$ is a non decreasing sequence of positive integers. If $k(n) = n$ then $M_n^{(1)} = Y^*_{k(n)}$ and $M_n^{(2)} = S^*_{(k)n}$

Thus $Y^*_{k(n)}$ and $S^*_{(k)n}$ may be regarded as general forms of $M_n^{(1)}$ and $M_n^{(2)}$ respectively.

Next we make the following assumptions on k(n).

k(n) is non-decreasing

$$\sup \left[k(n+1) - k(n) \right] \leq \mu \quad \text{(finite)}$$

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and

$$k(n)=\left[\frac{n}{(\log\ n)^{t(n)}}\right],\quad t(n)\to p, 0\le p\le\infty, \text{as } n\to\infty$$

in particular, if

$$k(n)=\begin{cases}[\alpha\ n], 0<\alpha\le 1 & \text{then} \quad p = 0\\ \left[\frac{n}{(\log\ n)^{q}}\right], 0<q<\infty, & \text{then} \quad p = q\\ [n^{\alpha}], 0<\alpha<1, & \text{then} \quad p = \infty\end{cases}$$

now define,

$$a(n) = \frac{\log\ \log\ n}{-\log(1-a)},\ b(n) = \frac{\log n}{-\log(1-a)}$$

Then we establish the following.
When $0\le p<\infty$, Nagesh S.(2014) establishes the result,

Theorem 1.1:
The almost sure limit set of the vector sequence $\{(Y^*_{k(n)}\text{-}b(n))/a(n), (S^*_{k(n)}\text{-}b(n))/a(n)\}$, $n\ge 1$ coincides with the region,
$S = \{(x,y) : r\text{-}p\text{-}1 \le x \le r, r\text{-}p\text{-}1 \le y \le r,\ x \ge y, x+y \le 2r\text{-}p\text{-}1\}$.
A generalization of this result when $p = \infty$, is presented in the next section.

ALMOST SURE LIMIT SET RESULT AND PROOF

Theorem 2.1:
The almost sure limit set of the vector sequence

$$\left\{\frac{Y^*_{k(n)}}{b(n)}, \frac{S^*_{k(n)}}{b(n)}\right\}\ n\ge 1 \text{ coincides with the region,}$$

$S = \{(x, y):\ \Delta \le x,\ \Delta \le y \text{ and } x + y \le 1 +\Delta\}$,
when $0 \le \Delta < 1$.
For proving the theorem we make use of the following Lemmas.

Lemma 2.1.1
For every $\varepsilon > 0$, $x > y > \Delta$ and $x+y \le 1 +\Delta$

$$P\left\{ Y^*_{k(l_i)} > c_{l_i}(x+\varepsilon),\ S^*_{k(l_i)} > c_{l_i}(y) \quad i.o. \right\} = 0$$

and

$$P\left\{ Y^*_{k(l_i)} > c_{l_i}(x),\ S^*_{k(l_i)} > c_{l_i}(y+\varepsilon) \quad i.o. \right\} = 0$$

where $l_i = [i^{\theta}]$ with $\theta = 1/(x+y-2\Delta+\varepsilon/2)$
and $c_{l_i}(x) = x\, b(l_i)$

Proof:
We have

$$P\left\{ Y^*_{k(l_i)} > c_{l_i}(x+\varepsilon),\ S^*_{k(l_i)} > c_{l_i}(y) \right\}$$
$$=P\left\{ Y^*_{k(l_i)} > c_{l_i}(x+\varepsilon) \right\} - P\left\{ Y^*_{k(l_i)} > c_{l_i}(x+\varepsilon),\ S^*_{k(l_i)} \le c_{l_i}(y) \right\}$$

$$= \left\{ 1-F^{k(l_i)} (c_{l_i}(x+\varepsilon)) \right\} - k(l_i) \left\{ F^{k(l_i)-1} (c_{l_i}(y)) \right\} \left\{ 1-F [c_{l_i}(x+\varepsilon)) \right\}$$

$$=1-\exp \left\{ -k(l_i) \left[1-F(c_{l_i}(x+\varepsilon)) \right] (1+o(1)) \right\} -k(l_i) \left[\exp\{-(k(l_i)-1) \right.$$

$$\left[1-F(c_{l_i}(y)) \right] (1+o(1)) \left] \left[1-F (c_{l_i}(x+\varepsilon)) \right] \right.$$

$$= 1+o(1) \left\{ k(l_i)(k(l_i)-1) \right\} \left[1-F(c_{l_i}(x+\varepsilon)) \right] \left[1-F (c_{l_i}(y)) \right],$$

Since $k(l_i) \left[1-F(c_{l_i}(x+\varepsilon)) \right] \to 0$ for $x > \Delta$,

$k(l_i) \left[1-F(c_{l_i}(y)) \right] \to 0$ for $y > \Delta$,

and

$$\frac{[1-F(c_{l_i}(x+\varepsilon))]}{[1-F(c_{l_i}(y))]} \to 0 \text{ for } x > y \text{ as } i \to \infty$$

$$\leq \text{Const. } k^2(l_i) \left[1-F(c_{l_i}(x+\varepsilon)) \right] \left[1-F (c_{l_i}(y)) \right]$$

for all i large

$$\leq \text{Const} \left(\frac{1}{l_i^{(x+\varepsilon-\Delta+\eta)}} \right) \left(\frac{1}{l_i^{(y-\Delta+\eta)}} \right), \eta > 0$$

$$= \text{Const} \left(\frac{1}{i^{\theta(x+\varepsilon-\Delta+\eta+y-\Delta+\eta)}} \right)$$

$$= \text{Const} \left(\frac{1}{i^{\theta(x+y+\varepsilon-2\Delta-2\eta}} \right)$$

$$= \text{Const} \left(\frac{1}{i^{1+\eta_7}} \right), \ \eta_7 > 0.$$

Thus the proof of the Lemma is complete. .

Lemma 2.1.2

For every $\varepsilon > 0$ and $x_0 = \Delta - \varepsilon$,

$$P \left\{ S^*_{k(n)} \leq c_n(x_0) \text{ i. o. } \right\} = 0.$$

Proof:

In view of Lemma Barndorff-Neilson(1961) and Devroye (1981) it is sufficient to prove,

$$P \left\{ S^*_{k(n)} \leq c_n(x_0) \right\} \to 0$$

and $$\sum_{n=1}^{\infty} P \left\{ S^*_{k(n)} \leq c_n(x_0) \text{ and } S^*_{k(n+1)} > c_n(x_0) \right\} < \infty$$

Note that

$$P \left\{ S^*_{k(n)} \leq c_n(x_0) \right\} = P \left\{ S^*_{k(n)} \leq c_n(x_0),\ Y^*_{k(n)} \leq c_n(x_0) \right\} + P \left\{ S^*_{k(n)} \leq c_n(x_0),\ Y^*_{k(n)} > c_n(x_0) \right\}$$

$$= P \left\{ Y^*_{k(n)} \leq c_n(x_0) \right\} + P \left\{ S^*_{k(n)} \leq c_n(x_0),\ Y^*_{k(n)} > c_n(x_0) \right\}$$

$$= F^{k(n)} (c_n(x_0)) + k(n) \left[F^{k(n)-1} (c_n(x_0) \right] \left[1-F(c_n(x_0)) \right]$$

$$= E_n \left[F^{k(n)-1} (c_n(x_0) \right] \left[1 + \frac{F(c_n(x_0))}{E_n} \right],$$

where $E_n = k(n) \left\{ 1 - F(c_n(x_0)) \right\}$

$$\le \left(\frac{1}{n^{(x_o - \Delta - \eta)}} \right) \left(\exp\left\{ -k(n)\left[1 - F(c_n(x_0)) \right](1+o(1)\right\} \right)$$

$$\le \left(\frac{1}{n^{(x_o - \Delta - \eta)}} \right) \left(\exp\left\{ \frac{1}{n^{(x_o - \Delta - \eta)}} \right\} \right)$$

$$= \left(n^{3\varepsilon/2} (\exp(-n^{\varepsilon/2})) \right)$$

$\rightarrow 0$, as $n \rightarrow \infty$.

Now since $\{c_n(x_0)\}$ is non decreasing, notice that

$$P \left\{ S^*_{k(n)} \le c_n(x_0) \text{ and } S^*_{k(n+1)} > c_n(x_0) \right\}$$
$$\le P \left\{ S^*_{k(n)} \le c_n(x_0),\ Y^*_{k(n)} \le c_n(x_0),\ S^*_{k(n+1)} > c_n(x_0) \right\}$$
$$+ P \left\{ S^*_{k(n)} \le c_n(x_0),\ Y^*_{k(n)} > c_n(x_0),\ S^*_{k(n+1)} > c_n(x_0) \right\} \quad (2.1.1)$$

Case(i):

$n-k(n+1)+2 \ge n - k(n)+1$

RHS of (2.1.1)

$\le k(n) [F^{k(n)-1}(c_n(x_0))] [1-F(c_n(x_0))]^2$

$= E_n [F^{k(n)-1}(c_n(x_0))][1-F(c_n(x_0))]$

$$\le n^{3\varepsilon/2} \left(\exp\{-n^{\varepsilon/2}\} \right) \left(\frac{1}{n^{x_o}(\log n)^{-(r-1)}} \right)$$

$$= \left(\frac{1}{n^{(\Delta - \varepsilon - 3\varepsilon/2)}} \right) \left(\frac{1}{(\log n)^{-(r-1)}} \right) \left(\exp\{-n^{\varepsilon/2}\} \right)$$

$$\le \frac{\text{Const.}}{n^{1+\eta_8}} \quad \eta_8 > 0.$$

Case(ii):

$n - k(n+1) - 2 < n - k(n) + 1.$

Then proceeding as in the proof of Lemma 2.5, in Nagesh S.(2014),

R.H.S. of (2.1.1) is majorised by

$$\left(\frac{1}{n^{1+\eta_9}} \right), \eta_9 > 0.$$

Thus the proof of the Lemma is complete.

Lemma 2.1.3 :

For $\Delta \le x < 1+\Delta$, $\Delta \le y < 1+\Delta$, $x > y$ and $x+y \le 1 + \Delta$,

$$P \left\{ Y^*_{k(l_i)} > c_{l_i}(x),\ S^*_{k(l_i)} > c_{l_i}(y) \text{ i. o.} \right\} = 1$$

Proof:

We have

$$P \left\{ Y^*_{k(l_i)} > c_{l_i}(x),\ S^*_{k(l_i)} > c_{l_i}(y) \right\}$$
$$= P \left\{ Y^*_{k(l_i)} > c_{l_i}(x) \right\} - P \left\{ Y^*_{k(l_i)} > c_{l_i}(x),\ S^*_{k(l_i)} \le c_{l_i}(y) \right\}$$
$$= \left[1 - F^{k(l_i)}(c_{l_i}(x)) \right] - \left[k(l_i) \left[F^{k(l_i)-1}(c_{l_i}(y)) \right] \left[1 - F(c_{l_i}(x)) \right] \right]$$
$$= 1 - \exp\left\{ -k(l_i)\left[1 - F(c_{l_i}(x)) \right](1 + o(1)) \right\} - k(l_i) \left\{ \exp \right.$$

$(-k(l_i)-1)\left[1-F(c_{l_i}(y))\right](1+o(1))\}\left[1-F(c_{l_i}(x))\right]$

$=(1+o(1))\left[(k(l_i)(k(l_i)-1)\right]\left[1-F(c_{l_i}(x)\right]\left[1-F(c_{l_i}(y))\right]$

$\geq \text{Const}.\,k^2(l_i)\left[1-F(c_{l_i}(x))\right]\left[1-F(c_{l_i}(y))\right]$ for all i large

$\geq$ Const. $\left(\frac{1}{l_i^{(x-\Delta=\eta)}}\right)\left(\frac{1}{l_i^{(y-\Delta=\eta)}}\right)$, $\eta > 0$

= Const. $\left(\frac{1}{l_i^{(x+y-2\Delta+2\eta)}}\right)$

= Const $\left(\frac{1}{i^{\theta(x+y-2\Delta+2\eta)}}\right)$

= Const $\left(\frac{1}{i^{1-\eta_{10}}}\right)$ $\eta_{10} > 0$,

Now we have to show that

$l_{i+1} - k(l_{i+1}) + 1 - l_i > 0$

$$l_{i+1} - k(l_{i+1})+1- l_i = l_{i+1}\left[1-\frac{k(l_{i+1})}{l_{i+1}}+\frac{1}{l_{i+1}}+\frac{l_i}{l_{i+1}}\right] \quad (2.1.2)$$

$$\frac{l_i}{l_{i+1}} = \frac{i^\theta}{(i+1)^\theta} = \frac{i^\theta}{i^\theta\left(1+\frac{1}{i}\right)^\theta} = \left(1+\frac{1}{i}\right)^{-\theta} = 1-\frac{\theta}{i}(1+o(1))$$

Therefore,

$$1-\frac{l_i}{l_{i+1}} = \frac{\theta}{i}(1+o(1))$$

$$1-\frac{l_i}{l_{i+1}} - \frac{k(l_{i+1})}{l_{i+1}} = \frac{\theta}{i}(1+o(1)) - \frac{k(l_{i+1})}{l_{i+1}}$$

$$= \frac{\theta}{i}\left[1+o(1)) - \frac{i}{\theta}\frac{k(l_{i+1})}{l_{i+1}}\right]$$

$$= i\,\frac{k(l_{i+1})}{l_{i+1}} \to 0,\ \text{iff}\ \theta > \frac{1}{1-\Delta}$$

Therefore R.H.S. of (2.1.2) is $= \frac{(i+1)^\theta}{i}(1+o(1)) = i^{\theta-1} \to \infty$ as $\theta > 1$

Thus the events $\left\{ Y^*_{k(l_i)} > c_{l_i}(x),\ S^*_{k(l_i)} > c_{l_i}(y) \right\}$ i=1,2, …, are independent for all i large. Hence by Borel-Cantell; Lemma, the required result is obtained.

Lemma 2.1.4:

For all $x \geq \Delta$, $y \geq \Delta$ with $x+y > \Delta+1$ and for every $\varepsilon>0$,

$$P\left\{ Y^*_{k(n)} > c_n(x+\varepsilon),\ S^*_{k(n)} > c_n(y+\varepsilon)\ \text{i.o.} \right\} = 0$$

Proof:

Define the events

$$A_n = \left\{ Y^*_{k(n)} > c_n(x+\varepsilon)\ ,\ S^*_{k(n)} > c_n(y+\varepsilon) \right\}$$

and

$$B_i = \left\{ Y^*_{k(n)} > c_{n_i}(x+\varepsilon)\ ,\ S^*_{k(n)} > c_{n_i}(y+\varepsilon) \text{ for atleast one } n \in [n_i, n_{i+1}] \right\}$$

Where $n_i = [i^\theta]$, $i > 1$ and $\theta = \left(\dfrac{1}{x + y - 2\Delta + \varepsilon/2} \right)$

Notice that

$P\{A_n \text{ i.o. in } n\} \le P(B_i \text{ i. o. in } i\}$ (2.1.3)

But $P(B_i) \le P\{\max(X_{n_i - k(n_i)+1}, \ldots, X_{n_{i+1}}) > c_{n_i}(x+\varepsilon),$

second max $(X_{n_i - k(n_i)+1}, \ldots, X_{n_{i+1}}) > c_{n_i}(y+\varepsilon)\}$

$$\le \left[n_{i+1} - n_i + k(n_i) - 1\right]^2 \left[1 - F(c_{n_i}(x+\varepsilon))\right] \left[1 - F(c_{n_i}(y+\varepsilon))\right]$$

$$= \left[\frac{n_{i+1} - n_i + k(n_i) - 1}{n_i}\right]^2 n_i \left[1 - F(c_{n_i}(x+\varepsilon))\right] n_i \left[1 - F(c_{n_i}(y+\varepsilon))\right]$$

$$\le \text{Const} \left(\frac{1}{i^{1+\eta_{11}}} \right) \quad \eta_{11} > 0,$$

By Borel-Cantelli Lemma, R.H.S. of (2.1.3) is zero and thus the Lemma is proved.

PROOF OF THE THEOREM

By Lemmas 2.1.1, 2.1.2, 2.1.3, 2.1.4, it follows that the required limit set is contained in S. This completes the proof of the theorem.

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