Original Article

Expression for primitive idempotents in FC_{16p}

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Abstract The explicit expressions for complete set of primitive idempotents in FC_{16p^n}, the group algebra of the cyclic group C_{16p^n} of order $16p^n$ over the field F of prime power order q of the form $16k+9$, are obtained.

Keywords: Group algebra, cyclotomic cosets, primitive idempotents.

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INTRODUCTION

Let F be a finite field of prime power order q and G be a cyclic group of order m such that $g.c.d.(m,q)=1$. Then FG, the group algebra of the cyclic group G over F, is semi-simple and has only a finite number of primitive idempotents which equals the number of cyclotomic cosets modulo m. Let t be the multiplicative order of q modulo m, then $1 \le t \le \phi(m)$ [3]. If $t=\phi(m)$ and $m=2,4,p^n,2p^n$, the complete sets of primitive idempotents were calculated by Pruthi and Arora [1, 4]. The minimal cyclic codes of length $p^n q$ were discussed by Bakshi and Raka[2]. Primitive idempotents of length $4p^n$ were obtained by Chawla and Singh[5] and those of length $8p^n$ were discussed by Singh and Arora[6].

The q-cyclotomic cosets modulo $16p^n$,where $p^n \equiv 1(\mathrm{mod}\,8)$ and q is of the form $16k+9$,are obtained in Section 2. In Section 3, we obtained the primitive idempotents in FC_{16p^n} (Theorem 3.19).

CYCLOTOMIC COSETS

Let $S=\{1,2,...,16p^n\}$. For $a,b \in S$, say that $a \square b$ iff $a \equiv bq^i(\mathrm{mod}\,16p^n)$ for some integer $i \ge 0$. This is an equivalence relation on set S. The equivalence classes due to this relation are called q - cyclotomic cosets modulo

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$16p^n$. The q-cyclotomic coset containing $s \in S$ is $\Omega_s = \{s, sq, sq^2, ..., sq^{t_s-1}\}$, where t_s is the smallest positive integer such that $sq^{t_s} \equiv s (\bmod 16p^n)$.

Lemma 2.1. If $\phi(p^n)$ is the order of q modulo p^n then order of q modulo p^{n-i} is $\phi(p^{n-i})$, for all i, $0 \le i \le n-1$.
Proof is trivial.

Lemma 2.2. If q is an odd prime of the form $16k+9$ and $\phi(p^n)$ is the order of q modulo p^n, then for $0 \le i \le n-1$, the order of q modulo $2p^{n-i}$, $4p^{n-i}$, $8p^{n-i}$ and $16p^{n-i}$ is $\phi(p^{n-i})$.

Proof. Since $\phi(p^n)$ is the order of q modulo p^n, therefore, by lemma 2.1, order of q modulo p^{n-i} is $\phi(p^{n-i})$, $0 \le i \le n-1$. Hence

$$q^{\phi(p^{n-i})} \equiv 1 \ (\bmod\ p^{n-i}) \qquad (1)$$

Since $\phi(p^{n-i})$ is even, therefore, $\varphi(p^{n-i}) = 2t$ for some integer t and $q = 16k+9$, for some integer k, therefore, $q \equiv 1 \ (\bmod\ 2)$. Hence $q^{\phi(p^{n-i})} \equiv 1 \ (\bmod\ 2)$

Also, $g.c.d.(2, p^{n-i}) = 1$, therefore, $q^{\phi(p^{n-i})} \equiv 1 \ (\bmod\ 2p^{n-i})$.

As order of q modulo p^{n-i} is $\phi(p^{n-i})$, therefore, $\phi(p^{n-i})$ is the smallest integer for which (1) holds. Hence the order of q modulo $2p^{n-i}$ is $\phi(p^{n-i})$. Similarly, the result holds for $4p^{n-i}$, $8p^{n-i}$ and $16p^{n-i}$.

Lemma 2.3. For $0 \le i \le n-1$ and $0 \le k \le \phi(p^{n-i})-1$, $2^r\left(1+2sp^n\right) \not\equiv q^k \ (\bmod 16p^{n-i})$, for $0 \le r \le 2$ and $1 \le s \le 7$
.

Equivalently, $p^i(1+2sp^n) \not\equiv p^i q^k \ (\bmod 16p^n)$.
Proof can be obtained using lemma 2.1 - 2.2.

Theorem 2.4. The q-cyclotomic cosets modulo $16p^n$ are

$\Omega_{lp^n} = \left\{lp^n\right\}$ for $l = 0,2,4,6,8,10,12,14$;

$\Omega_{sp^n} = \left\{sp^n, sp^n q\right\}$, for $s = 1, \lambda, \mu, \nu$;

and $\Omega_{tp^j} = \{tp^j, tp^j q, ..., tp^j q^{\phi(p^{n-j})-1}\}$, for $0 \le j \le n-1$,

$t = 1,2,4,8, \lambda = 1+2p^n, 2\lambda, 4\lambda, \mu = 1+4p^n, 2\mu, \nu = 1+6p^n, 2\nu, \chi = 1+8p^n, \psi = 1+10p^n,$
$\xi = 1+12p^n, \tau = 1+14p^n.$

Proof. $\Omega_0 = \left\{0\right\}$ is trivial.

Since q is odd prime of the form $16k+9$, so $q^2 \equiv 1\left(\bmod 16\right)$, therefore, $sp^n q^2 \equiv sp^n \left(\bmod 8p^n\right)$. Hence $\Omega_{sp^n} = \left\{sp^n, sp^n q\right\}$.

Since $q \equiv 1\left(\bmod 8\right)$, therefore, $lp^n q \equiv lp^n \left(\bmod 16p^n\right)$. Hence $\Omega_{lp^n} = \left\{lp^n\right\}$.

By lemma 2.2, $q^{\phi(p^{n-i})} \equiv 1\left(\bmod 16p^{n-i}\right)$, equivalently, $p^i q^{\phi(p^{n-i})} \equiv p^i \left(\bmod 16p^n\right)$.

Therefore, $\Omega_{p^i} = \{p^i, p^i q, ..., p^i q^{\phi(p^{n-i})-1}\}$.

Similarly, $\Omega_{tp^i} = \{tp^i, tp^i q, ..., tp^i q^{\phi(p^{n-i})-1}\}$.

Obviously, $|\Omega_{lp^n}|=1$ for $l=0,2,4,6,8,10,12,14$; $|\Omega_{sp^n}|=2$ for $s=1,\lambda,\mu,\nu$.

$|\Omega_{tp^i}|=\phi(p^{n-i})$ for $0\le j\le n-1$,

$t=1,2,4,8,\lambda=1+2p^n,2\lambda,4\lambda,\mu=1+4p^n,2\mu,\nu=1+6p^n,2\nu,\chi=1+8p^n,\psi=1+10p^n,$

$\xi=1+12p^n,\tau=1+14p^n.$

Therefore, $\sum_{i=0}^{n-1}\left|\Omega_{p^i}\right|=\sum_{i=0}^{n-1}\phi(p^{n-i})=\varphi(p^n)+\varphi(p^{n-1})+\varphi(p^{n-2})+...+\varphi(p)=p^n-1.$

Hence $|\Omega_0|+|\Omega_{p^n}|+|\Omega_{2p^n}|+|\Omega_{4p^n}|+|\Omega_{6p^n}|+|\Omega_{8p^n}|+|\Omega_{10p^n}|+|\Omega_{12p^n}|+|\Omega_{14p^n}|$

$$+|\Omega_{\lambda p^n}|+|\Omega_{\mu p^n}|+|\Omega_{\nu p^n}|+\sum_{i=0}^{n-1}\left\{\sum_{t=1,2,4,8,16,\lambda,2\lambda,4\lambda,\mu,2\mu,\nu,2\nu,\chi,\psi,\xi,\tau}|\Omega_{tp^i}|\right\}=8p^n.$$

Thus, it follows that these are the only distinct q - cyclotomic cosets modulo $16p^n$.

PRIMITIVE IDEMPOTENTS

3.1 Notation. Throughout this paper, we assume that α is a $16p^n$ th root of unity in some extension field of F. We denote $\theta_s(x)$, the primitive idempotent in FC_{16p^n}, given by $\theta_s(x)=\frac{1}{16p^n}\sum_{i=0}^{16p^n-1}\varepsilon_i^s x^i$ where $\varepsilon_i^s=\sum_{j\in\Omega_s}\alpha^{-ij}$ [3]. Also, denote $\overline{C}_s=\sum_{s\in\Omega_s}x^s$.

3.2 Lemma. For any odd prime p and positive integer k, if δ is a primitive p^k th root of unity and ζ is a primitive $2p^k$ th root of unity in some extension field of F. Let q be a primitive root modulo p^k, then

$$\sum_{s=0}^{\phi(p^k)-1}\delta^{q^s}=\begin{cases}-1 & \text{if } k=1\\ 0 & \text{if } k>1\end{cases}$$

and
$$\sum_{s=0}^{\phi(2p^k)-1}\zeta^{q^s}=\begin{cases}1 & \text{if } k=1\\ 0 & \text{if } k>1.[6]\end{cases}$$

3.3 Lemma. If $p^n\equiv 1(\mathrm{mod}\,8)$, then $\left(1+14p^n\right)q^{\frac{\phi(p^n)}{2}}\equiv -1\left(\mathrm{mod}\,16p^n\right)$.

Proof. By lemma 2.1, $q^{\frac{\phi(p^n)}{2}}\equiv -1(\mathrm{mod}\,p^n)$. Also, $1+14p^n\equiv 1(\mathrm{mod}\,p^n)$. Therefore, $\left(1+14p^n\right)q^{\frac{\phi(p^n)}{2}}\equiv -1(\mathrm{mod}\,p^n)$. Further, since $p^n\equiv 1(\mathrm{mod}\,8)$. Also q is of the form $16k+1$, therefore $q^{\frac{\phi(p^n)}{2}}$ is of the form $16s+1$. Let $p^n=8k+1$, then $\left(1+14p^n\right)q^{\frac{\phi(p^n)}{2}}\equiv -1\left(\mathrm{mod}\,16\right)$. But $g.c.d(16,p^n)=1$, thus $\left(1+14p^n\right)q^{\frac{\phi(p^n)}{2}}\equiv -1\left(\mathrm{mod}\,16p^n\right)$

3.4 Remark. In above lemma, we obtained "If $p^n\equiv 1(\mathrm{mod}\,8)$, then $-\Omega_1=\Omega_\tau$ ".

3.5 Lemma. For cyclotomic cosets Ω_{p^j}, $0\le j\le n-1$,

(i) $\nu^2\Omega_{p^j}=\Omega_{p^j}=\nu\Omega_{\nu p^j}$

(ii) $\tau^2\Omega_{p^j}=\Omega_{p^j}=\tau\Omega_{\tau p^j}$

(iii) $\chi^2\Omega_{p^j} = \Omega_{p^j} = \chi\Omega_{\chi p^j}$.

Proof. (i) Since $\nu^2 = (1+6p^n)^2 = 1+36p^{2n}+12p^n = 1+4p^n(9p^n+3)$.

As $p^n \equiv 1(\bmod 4)$, so $(9p^n+3) \equiv 0(\bmod 4)$ and hence $\nu^2 \equiv 1(\bmod 16p^n)$.

Therefore, $\lambda^2\Omega_{p^j} = \Omega_{p^j}$. Also,

$$\lambda^2\Omega_{p^j} = \left\{\lambda^2 p^j, \lambda^2 p^j q, ..., \lambda^2 p^j q^{\phi(p^{n-j})-1}\right\} = \lambda\left\{\lambda p^j, \lambda p^j q, ..., \lambda p^j q^{\phi(p^{n-j})-1}\right\} = \lambda\Omega_{\lambda p^j}.$$

Thus, $\lambda^2\Omega_{p^j} = \Omega_{p^j} = \lambda\Omega_{\lambda p^j}$.

Proof of remaining parts will go on similar lines.

3.6 Lemma. For $0 \le j \le n-1$,

(i) $-\Omega_{2p^j} = \Omega_{2\nu p^j}$.

(ii) $-\Omega_{4p^j} = \Omega_{4p^j}$.

(iii) $-\Omega_{8p^j} = \Omega_{8p^j}$

(iv) $-\Omega_{16p^j} = \Omega_{16p^j}$

Proof of these can be obtained easily by using lemma 3.3.

3.7 Lemma. For cyclotomic cosets Ω_{sp^n} for $s = 1,2,4,6,8,10,12,14,\lambda,\mu,\nu$

(i) $-\Omega_{p^n} = \Omega_{\nu p^n}$, $-\Omega_{\lambda p^n} = \Omega_{\mu p^n}$.

(ii) $-\Omega_{2p^n} = \Omega_{14p^n}$, $-\Omega_{4p^n} = \Omega_{12p^n}$, $-\Omega_{6p^n} = \Omega_{10p^n}$, $-\Omega_{8p^n} = \Omega_{8p^n}$

Proof. Since $\Omega_{p^n} = \{p^n, p^n q\}, \Omega_{\nu p^n} = \{\nu p^n, \nu p^n q\}$, and $p^n \equiv 1(\bmod 4)$ so let $p^n = 4k'+1$. Then $\nu p^n q + p^n = (\nu q+1)p^n = \{8k+5+3p^n q\}2p^n$ and hence

$$\nu p^n q \equiv -p^n(\bmod 16p^n)$$

Hence the result follows.

3.8 Lemma. If $r = 1,2,4,8,16,\lambda,\mu,\nu,\chi,\psi,\xi$ or τ then for $0 \le j \le n-1$,

$$r\Omega_{16p^j} = \Omega_{16p^j} = 16\Omega_{rp^j}.[6]$$

3.9 Notations. Let α be a fixed primitive $16p^n$ th root of unity in some extension field of F. For $0 \le j \le n-1$, we define, $T_j = p^j \sum_{s\in\Omega_{p^j}} \alpha^s$, $P_j = p^j \sum_{s\in\Omega_{\tau p^j}} \alpha^s$, $V_j = p^j \sum_{s\in\Omega_{\xi p^j}} \alpha^s$, $X_j = p^j \sum_{s\in\Omega_{\psi p^j}} \alpha^s$, $R_j = p^j \sum_{s\in\Omega_{2p^j}} \alpha^s$, $M_j = p^j \sum_{s\in\Omega_{4p^j}} \alpha^s$ and $Y_j = p^j \sum_{s\in\Omega_{2\lambda p^j}} \alpha^s$. Then, $T_j, P_j, V_j, R_j, M_j, Y_j \in F$.

3.10 Lemma. For $0 \le j, i \le n$,

$$\sum_{s\in\Omega_{p^j}}\alpha^{p^i s}=\sum_{s\in\Omega_{\nu p^j}}\alpha^{\nu p^i s}=\sum_{s\in\Omega_{\tau p^j}}\alpha^{\tau p^i s}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{\chi p^i s}=\sum_{s\in\Omega_{\psi p^j}}\alpha^{\psi p^i s}=-\sum_{s\in\Omega_{\mu p^j}}\alpha^{\mu p^i s}=-\sum_{s\in\Omega_{\xi p^j}}\alpha^{\xi p^i s}$$

$$=\sum_{s\in\Omega_{\mu p^j}}\alpha^{\xi p^i s}=\begin{cases}0 & \text{if } i+j\geq n\\ \frac{1}{p^j}T_{i+j} & \text{if } i+j\leq n\text{-}1\end{cases}$$

Proof. Let $\delta=\alpha^{p^{i+j}}$. Then, $\sum_{s\in\Omega_{p^j}}\alpha^{p^i s}=\sum_{t=0}^{\phi(p^{n-j})-1}\alpha^{p^{i+j}q^t}=\sum_{t=0}^{\phi(p^{n-j})-1}\delta^{q^t}$.

Now consider the following cases:

Case 1. If $i+j\geq n$, then δ is 16th root of unity and

$$\delta^{q^l}=\delta^{q^r} \text{ iff } q^l\equiv q^r\left(\text{mod}16\right) \text{ iff } l\equiv r\left(\text{mod}\,2\right).$$

Hence using lemma 3.2, $\sum_{s\in\Omega_{p^j}}\alpha^{p^i s}=\frac{\phi\left(p^{n-j}\right)}{2}\sum_{t=0}^{1}\delta^{q^t}=0\cdot$

Case 2. If $i+j\leq n-1$, then δ is $16\,p^{n-i-j}$ th root of unity and

$$\delta^{q^l}=\delta^{q^r} \text{ iff } q^l\equiv q^r\left(\text{mod}16p^{n-i-j}\right) \text{ iff } l\equiv r\left(\text{mod}\,\phi\left(p^{n-i-j}\right)\right)$$

Therefore, $\sum_{s\in\Omega_{p^j}}\alpha^{p^i s}=\frac{\phi\left(p^{n-j}\right)}{\phi\left(p^{n-i-j}\right)}\sum_{t=0}^{\phi(p^{n-i-j})-1}\delta^{q^t}=\frac{1}{p^j}p^{i+j}\sum_{t=0}^{\phi(p^{n-i-j})-1}\alpha^{p^{i+j}q^t}=\frac{1}{p^j}p^{i+j}\sum_{s\in\Omega_{p^{i+j}}}\alpha^s$

Since $p^{i+j}\sum_{s\in\Omega_{p^{i+j}}}\alpha^s=T_{i+j}$, therefore, $\sum_{s\in\Omega_{p^j}}\alpha^{p^i s}=\frac{1}{p^j}T_{i+j}\cdot$

The proof of the following lemmas will go on similar lines as of the lemma 3.10.

3.11 Lemma. For $0\leq j\leq n-1, 0\leq i\leq n$,

$$\sum_{s\in\Omega_{\nu p^j}}\alpha^{p^i s}=-\sum_{s\in\Omega_{\nu p^j}}\alpha^{\chi p^i s}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{\tau p^i s}=-\sum_{s\in\Omega_{p^j}}\alpha^{\tau p^i s}=-\sum_{s\in\Omega_{\mu p^j}}\alpha^{\lambda p^i s}=-\sum_{s\in\Omega_{\lambda p^j}}\alpha^{\chi p^i s}$$

$$=-\sum_{s\in\Omega_{\psi p^j}}\alpha^{\xi p^i s}=\begin{cases}0 & \text{if } i+j\geq n\\ -\frac{1}{p^j}P_{i+j} & \text{if } i+j\leq n\text{-}1\end{cases}$$

3.12 Lemma. For $0\leq j,i\leq n$,

$$\sum_{s\in\Omega_{\mu p^j}}\alpha^{p^i s}=-\sum_{s\in\Omega_{\mu p^j}}\alpha^{\chi p^i s}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{\xi p^i s}=-\sum_{s\in\Omega_{p^j}}\alpha^{\xi p^i s}=\begin{cases}0 & \text{if } i+j\geq n\\ -\frac{1}{p^j}V_{i+j} & \text{if } i+j\leq n\text{-}1\end{cases}$$

3.13 Lemma. For $0\leq j,i\leq n$,

$$\sum_{s\in\Omega_{\lambda p^j}}\alpha^{p^i s}=-\sum_{s\in\Omega_{\lambda p^j}}\alpha^{\chi p^i s}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{\psi p^i s}=-\sum_{s\in\Omega_{p^j}}\alpha^{\psi p^i s}=\sum_{s\in\Omega_{\mu p^j}}\alpha^{\nu p^i s}=-\sum_{s\in\Omega_{\mu p^j}}\alpha^{\tau p^i s}=-\sum_{s\in\Omega_{\xi p^j}}\alpha^{\nu p^i s}$$

$$=\sum_{s\in\Omega_{\xi p^j}}\alpha^{\tau p^i s}=\begin{cases}0 & \text{if } i+j\geq n\\ -\frac{1}{p^j}V_{i+j} & \text{if } i+j\leq n\text{-}1\end{cases}$$

3.14 Lemma. For $0\leq j\leq n-1, 0\leq i\leq n-1$,

$$\sum_{s\in\Omega_{p^j}}\alpha^{8p^is}=\sum_{s\in\Omega_{4p^j}}\alpha^{2p^is}=\sum_{s\in\Omega_{\lambda p^j}}\alpha^{8p^is}=\sum_{s\in\Omega_{\nu p^j}}\alpha^{8p^is}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{8p^is}=\sum_{s\in\Omega_{\mu p^j}}\alpha^{8p^is}$$

$$=\sum_{s\in\Omega_{\psi p^j}}\alpha^{8p^is}=\sum_{s\in\Omega_{8p^j}}\alpha^{\xi p^is}=\sum_{s\in\Omega_{8p^j}}\alpha^{\tau p^is}=\begin{cases}-\phi\left(p^{n-j}\right) & \text{if } i+j\geq n\\ \mathrm{p}^{n-j-1} & \text{if } i+j=n-1\\ 0 & \text{if } i+j<n-1\end{cases}$$

3.15 Lemma. For $0\leq j\leq n-1, 0\leq i\leq n-1$,

$$\sum_{s\in\Omega_{p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{8p^j}}\alpha^{2p^is}=\sum_{s\in\Omega_{4p^j}}\alpha^{4p^is}=\sum_{s\in\Omega_{\lambda p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{\nu p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{\mu p^j}}\alpha^{16p^is}$$

$$=\sum_{s\in\Omega_{\psi p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{8p^j}}\alpha^{16p^is}=\sum_{s\in\Omega_{8p^j}}\alpha^{16p^is}=\begin{cases}\phi\left(p^{n-j}\right) & \text{if } i+j\geq n\\ -\mathrm{p}^{n-j-1} & \text{if } i+j=n-1\\ 0 & \text{if } i+j<n-1\end{cases}$$

3.16 Lemma. For $0\leq i\leq n, 0\leq j\leq n-1$,

$$\sum_{s\in\Omega_{p^j}}\alpha^{2p^is}=-\sum_{s\in\Omega_{\mu p^j}}\alpha^{2p^is}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{2p^is}=-\sum_{s\in\Omega_{\chi p^j}}\alpha^{2p^is}=\begin{cases}\phi\left(p^{n-j}\right)\alpha^{2p^{i+j}} & \text{if } i+j\geq n, j<n\\ 2\alpha^{2p^{i+j}} & \text{if } i+j\geq n, j=n\\ \dfrac{1}{p^j}\mathrm{R}_{i+j} & \text{if } i+j\leq n-1\end{cases}$$

3.17 Lemma. For $0\leq j\leq n-1, 0\leq i\leq n-1$,

$$\sum_{s\in\Omega_{p^j}}\alpha^{4p^is}=-\sum_{s\in\Omega_{\lambda p^j}}\alpha^{4p^is}=-\sum_{s\in\Omega_{\nu p^j}}\alpha^{4p^is}=\sum_{s\in\Omega_{\chi p^j}}\alpha^{4p^is}=\sum_{s\in\Omega_{\mu p^j}}\alpha^{4p^is}$$

$$=-\sum_{s\in\Omega_{\psi p^j}}\alpha^{4p^is}=\sum_{s\in\Omega_{4p^j}}\alpha^{\xi p^is}=-\sum_{s\in\Omega_{4p^j}}\alpha^{\tau p^is}=\begin{cases}-\phi\left(p^{n-j}\right)\alpha^{4p^{i+j}} & \text{if } i+j\geq n, j<n\\ 2\alpha^{4p^{i+j}} & \text{if } i+j\geq n, j=n\\ \dfrac{1}{p^j}M_{i+j} & \text{if } i+j\leq n-1\end{cases}$$

3.18 Lemma. For $0\leq j,i\leq n$,

$$\sum_{s\in\Omega_{2\lambda p^j}}\alpha^{p^is}=\sum_{s\in\Omega_{2p^j}}\alpha^{\psi p^is}=-\sum_{s\in\Omega_{2p^j}}\alpha^{\nu p^is}=-\sum_{s\in\Omega_{2p^j}}\alpha^{\tau p^is}=\sum_{s\in\Omega_{2\mu p^j}}\alpha^{\tau p^is}=\begin{cases}0 & \text{if } i+j\geq n\\ \dfrac{1}{p^j}Y_{i+j} & \text{if } i+j\leq n-1\end{cases}$$

3.19 Theorem. The explicit expressions for the $16n+9$ primitive idempotents in FC_{16p^n} are given by

$$\theta_0(x)=\frac{1}{16p^n}\Big[\overline{C}_0+\overline{C}_{p^n}+\overline{C}_{2p^n}+\overline{C}_{4p^n}+\overline{C}_{6p^n}+\overline{C}_{8p^n}+\overline{C}_{10p^n}+\overline{C}_{12p^n}+\overline{C}_{14p^n}+\overline{C}_{\lambda p^n}+\overline{C}_{\mu p^n}+\overline{C}_{\nu p^n}$$
$$+\sum_{i=0}^{n-1}\Big\{\overline{C}_{p^i}+\overline{C}_{2p^i}+\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}+\overline{C}_{\lambda p^i}+\overline{C}_{2\lambda p^i}+\overline{C}_{4\lambda p^i}+\overline{C}_{\mu p^i}+\overline{C}_{2\mu p^i}+\overline{C}_{\nu p^i}+\overline{C}_{2\nu p^i}$$
$$+\overline{C}_{\chi p^i}+\overline{C}_{\psi p^i}+\overline{C}_{\xi p^i}+\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{p^n}\left(x\right)=\frac{1}{16p^n}\Big[2\overline{C}_0-2\gamma^{p^n}\overline{C}_{4p^n}+2\alpha^{6\nu p^{2n}}\overline{C}_{6p^n}-2\overline{C}_{8p^n}+2\gamma^{p^n}\overline{C}_{12p^n}-2\alpha^{6\nu p^{2n}}\overline{C}_{14p^n}$$
$$-\sum_{i=0}^{n-1}\left\{2\gamma^{p^i}\overline{C}_{4p^i}+2\overline{C}_{8p^i}-2\overline{C}_{16p^i}-2\beta^{p^i}\overline{C}_{2\lambda p^i}-2\gamma^{p^i}\overline{C}_{4\lambda p^i}-2\beta^{p^i}\overline{C}_{2\nu p^i}\right\}\Big]$$

$$\theta_{2p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0-\alpha^{6p^{2n}}\overline{C}_{p^n}-\gamma^{p^n}\overline{C}_{2p^n}-\overline{C}_{4p^n}+\gamma^{p^n}\overline{C}_{6p^n}+\overline{C}_{8p^n}-\gamma^{p^n}\overline{C}_{10p^n}-\overline{C}_{12p^n}+\gamma^{p^n}\overline{C}_{14p^n}$$
$$+\alpha^{6\lambda p^{2n}}\overline{C}_{\lambda p^n}+\alpha^{6p^{2n}}\overline{C}_{\mu p^n}-\alpha^{6\nu p^{2n}}\overline{C}_{\nu p^n}-\sum_{i=0}^{n-1}\Big\{\alpha^{6p^{n+i}}\overline{C}_{p^i}+\gamma^{p^i}\overline{C}_{2p^i}+\overline{C}_{4p^i}-\overline{C}_{8p^i}-\overline{C}_{16p^i}$$
$$-\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\lambda p^i}-\gamma^{p^i}\overline{C}_{2\lambda p^i}+\overline{C}_{4\lambda p^i}-\alpha^{6p^{n+i}}\overline{C}_{\mu p^i}+\gamma^{p^i}\overline{C}_{2\mu p^i}+\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\nu p^i}-\gamma^{p^i}\overline{C}_{2\nu p^i}$$
$$+\alpha^{6p^{n+i}}\overline{C}_{\chi p^i}-\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\psi p^i}-\alpha^{6p^{n+i}}\overline{C}_{\xi p^i}+\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{4p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0-\gamma^{p^n}\overline{C}_{p^n}-\overline{C}_{2p^n}+\overline{C}_{4p^n}-\overline{C}_{6p^n}+\overline{C}_{8p^n}-\overline{C}_{10p^n}+\overline{C}_{12p^n}-\overline{C}_{14p^n}-\gamma^{p^n}\overline{C}_{\lambda p^n}$$
$$+\gamma^{p^n}\overline{C}_{\mu p^n}-\gamma^{p^n}\overline{C}_{\nu p^n}-\sum_{i=0}^{n-1}\Big\{\gamma^{p^i}\overline{C}_{p^i}+\overline{C}_{2p^i}-\overline{C}_{4p^i}-\overline{C}_{8p^i}-\overline{C}_{16p^i}+\gamma^{p^i}\overline{C}_{\lambda p^i}+\overline{C}_{2\lambda p^i}-\overline{C}_{4\lambda p^i}$$
$$+\gamma^{p^i}\overline{C}_{\mu p^i}+\overline{C}_{2\mu p^i}-\gamma^{p^i}\overline{C}_{\nu p^i}+\overline{C}_{2\nu p^i}+\gamma^{p^i}\overline{C}_{\chi p^i}-\gamma^{p^i}\overline{C}_{\psi p^i}+\gamma^{p^i}\overline{C}_{\xi p^i}-\gamma^{p^i}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{6p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0-\beta^{p^n}\overline{C}_{p^n}+\gamma^{p^n}\overline{C}_{2p^n}-\overline{C}_{4p^n}-\gamma^{p^n}\overline{C}_{6p^n}+\overline{C}_{8p^n}+\gamma^{p^n}\overline{C}_{10p^n}-\overline{C}_{12p^n}-\gamma^{p^n}\overline{C}_{14p^n}$$
$$-\alpha^{2\lambda p^{2n}}\overline{C}_{\lambda p^n}+\alpha^{2p^{2n}}\overline{C}_{\mu p^n}-\alpha^{2\nu p^{2n}}\overline{C}_{\nu p^n}-\sum_{i=0}^{n-1}\Big\{\beta^{p^i}\overline{C}_{p^i}-\gamma^{p^i}\overline{C}_{2p^i}+\overline{C}_{4p^i}-\overline{C}_{8p^i}-\overline{C}_{16p^i}$$
$$+\alpha^{\left(2+4p^n\right)p^{n+i}}\overline{C}_{\lambda p^i}+\gamma^{p^i}\overline{C}_{2\lambda p^i}+\overline{C}_{4\lambda p^i}-\beta^{p^i}\overline{C}_{\mu p^i}-\gamma^{p^i}\overline{C}_{2\mu p^i}-\alpha^{\left(2+4p^n\right)p^{n+i}}\overline{C}_{\nu p^i}+\gamma^{p^i}\overline{C}_{2\nu p^i}$$
$$+\beta^{p^i}\overline{C}_{\chi p^i}+\alpha^{\left(2+4p^n\right)p^{n+i}}\overline{C}_{\psi p^i}-\beta^{p^i}\overline{C}_{\xi p^i}-\alpha^{\left(2+4p^n\right)p^{n+i}}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{8p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0-\overline{C}_{p^n}+\overline{C}_{2p^n}+\overline{C}_{4p^n}+\overline{C}_{6p^n}+\overline{C}_{8p^n}+\overline{C}_{10p^n}+\overline{C}_{12p^n}+\overline{C}_{14p^n}-\overline{C}_{\lambda p^n}-\overline{C}_{\mu p^n}-\overline{C}_{\nu p^n}$$
$$-\sum_{i=0}^{n-1}\Big\{\overline{C}_{p^i}-\overline{C}_{2p^i}-\overline{C}_{4p^i}-\overline{C}_{8p^i}-\overline{C}_{16p^i}+\overline{C}_{\lambda p^i}-\overline{C}_{2\lambda p^i}-\overline{C}_{4\lambda p^i}+\overline{C}_{\mu p^i}-\overline{C}_{2\mu p^i}+\overline{C}_{\nu p^i}-\overline{C}_{2\nu p^i}$$
$$+\overline{C}_{\chi p^i}+\overline{C}_{\psi p^i}+\overline{C}_{\xi p^i}+\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{10p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0+\alpha^{6p^{2n}}\overline{C}_{p^n}-\gamma^{p^n}\overline{C}_{2p^n}-\overline{C}_{4p^n}+\gamma^{p^n}\overline{C}_{6p^n}+\overline{C}_{8p^n}-\gamma^{p^n}\overline{C}_{10p^n}-\overline{C}_{12p^n}+\gamma^{p^n}\overline{C}_{14p^n}$$
$$+\alpha^{6\lambda p^{2n}}\overline{C}_{\lambda p^n}-\alpha^{6p^{2n}}\overline{C}_{\mu p^n}+\alpha^{6\nu p^{2n}}\overline{C}_{\nu p^n}+\sum_{i=0}^{n-1}\Big\{\alpha^{6p^{n+i}}\overline{C}_{p^i}-\gamma^{p^i}\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}$$
$$-\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\lambda p^i}+\gamma^{p^i}\overline{C}_{2\lambda p^i}-\overline{C}_{4\lambda p^i}-\alpha^{6p^{n+i}}\overline{C}_{\mu p^i}-\gamma^{p^i}\overline{C}_{2\mu p^i}+\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\nu p^i}+\gamma^{p^i}\overline{C}_{2\nu p^i}$$
$$+\alpha^{6p^{n+i}}\overline{C}_{\chi p^i}-\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\psi p^i}-\alpha^{6p^{n+i}}\overline{C}_{\xi p^i}+\alpha^{\left(6+4p^n\right)p^{n+i}}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{12p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0+\gamma^{p^n}\overline{C}_{p^n}-\overline{C}_{2p^n}+\overline{C}_{4p^n}-\overline{C}_{6p^n}+\overline{C}_{8p^n}-\overline{C}_{10p^n}+\overline{C}_{12p^n}-\overline{C}_{14p^n}-\gamma^{p^n}\overline{C}_{\lambda p^n}$$
$$+\gamma^{p^n}\overline{C}_{\mu p^n}-\gamma^{p^n}\overline{C}_{\nu p^n}+\sum_{i=0}^{n-1}\Big\{\gamma^{p^i}\overline{C}_{p^i}-\overline{C}_{2p^i}+\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}-\gamma^{p^i}\overline{C}_{\lambda p^i}-\overline{C}_{2\lambda p^i}+\overline{C}_{4\lambda p^i}$$
$$+\gamma^{p^i}\overline{C}_{\mu p^i}-\overline{C}_{2\mu p^i}-\gamma^{p^i}\overline{C}_{\nu p^i}-\overline{C}_{2\nu p^i}+\gamma^{p^i}\overline{C}_{\chi p^i}-\gamma^{p^i}\overline{C}_{\psi p^i}+\gamma^{p^i}\overline{C}_{\xi p^i}-\gamma^{p^i}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{14p^n}(x)=\frac{1}{16p^n}\Big[\overline{C}_0+\beta^{p^n}\overline{C}_{p^n}+\gamma^{p^n}\overline{C}_{2p^n}-\overline{C}_{4p^n}-\gamma^{p^n}\overline{C}_{6p^n}+\overline{C}_{8p^n}+\gamma^{p^n}\overline{C}_{10p^n}-\overline{C}_{12p^n}-\gamma^{p^n}\overline{C}_{14p^n}$$
$$+\alpha^{2\lambda p^{2n}}\overline{C}_{\lambda p^n}-\alpha^{2p^{2n}}\overline{C}_{\mu p^n}+\alpha^{2\nu p^{2n}}\overline{C}_{\nu p^n}+\sum_{i=0}^{n-1}\Big\{\beta^{p^i}\overline{C}_{p^i}+\gamma^{p^i}\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}$$
$$+\alpha^{(2+4p^n)p^{n+i}}\overline{C}_{\lambda p^i}-\gamma^{p^i}\overline{C}_{2\lambda p^i}-\overline{C}_{4\lambda p^i}-\beta^{p^i}\overline{C}_{\mu p^i}+\gamma^{p^i}\overline{C}_{2\mu p^i}-\alpha^{(2+4p^n)p^{n+i}}\overline{C}_{\nu p^i}-\gamma^{p^i}\overline{C}_{2\nu p^i}$$
$$+\beta^{p^i}\overline{C}_{\chi p^i}+\alpha^{(2+4p^n)p^{n+i}}\overline{C}_{\psi p^i}-\beta^{p^i}\overline{C}_{\xi p^i}-\alpha^{(2+4p^n)p^{n+i}}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{\lambda p^n}(x)=\frac{1}{16p^n}\Big[2\overline{C}_0-2\beta^{p^n}\overline{C}_{2p^n}+2\gamma^{p^n}\overline{C}_{4p^n}-2\alpha^{6p^{2n}}\overline{C}_{6p^n}-2\overline{C}_{8p^n}+2\beta^{p^n}\overline{C}_{10p^n}-2\gamma^{p^n}\overline{C}_{12p^n}$$
$$+2\alpha^{6p^{2n}}\overline{C}_{14p^n}-\sum_{i=0}^{n-1}\Big\{2\beta^{p^i}\overline{C}_{2p^i}-2\gamma^{p^i}\overline{C}_{4p^i}+2\overline{C}_{8p^i}-2\overline{C}_{16p^i}+2\gamma^{p^i}\overline{C}_{4\lambda p^i}+2\beta^{p^i}\overline{C}_{2\mu p^i}\Big\}\Big]$$

$$\theta_{\mu p^n}(x)=\frac{1}{16p^n}\Big[2\overline{C}_0-2\gamma^{p^n}\overline{C}_{4p^n}+2\alpha^{6\lambda p^{2n}}\overline{C}_{6p^n}-2\overline{C}_{8p^n}+2\gamma^{p^n}\overline{C}_{12p^n}-2\alpha^{6\lambda p^{2n}}\overline{C}_{14p^n}$$
$$+\sum_{i=0}^{n-1}\Big\{2\gamma^{p^i}\overline{C}_{4p^i}+2\overline{C}_{8p^i}-2\overline{C}_{16p^i}+2\beta^{p^i}\overline{C}_{2\lambda p^i}-2\gamma^{p^i}\overline{C}_{4\lambda p^i}-2\beta^{p^i}\overline{C}_{2\nu p^i}\Big\}\Big]$$

$$\theta_{\nu p^n}(x)=\frac{1}{16p^n}\Big[2\overline{C}_0+2\beta^{p^n}\overline{C}_{2p^n}+2\gamma^{p^n}\overline{C}_{4p^n}+2\alpha^{6p^{2n}}\overline{C}_{6p^n}-2\overline{C}_{8p^n}-2\beta^{p^n}\overline{C}_{10p^n}-2\gamma^{p^n}\overline{C}_{12p^n}$$
$$-2\alpha^{6p^{2n}}\overline{C}_{14p^n}+\sum_{i=0}^{n-1}\Big\{2\beta^{p^i}\overline{C}_{2p^i}+2\gamma^{p^i}\overline{C}_{4p^i}-2\overline{C}_{8p^i}+2\overline{C}_{16p^i}-2\gamma^{p^i}\overline{C}_{4\lambda p^i}-2\beta^{p^i}\overline{C}_{2\mu p^i}\Big\}\Big]$$

and for $0\le j\le n-1$,

$$\theta_{p^j}(x)=\frac{1}{16p^n}\Big[\phi(p^{n-j})\Big\{\overline{C}_0-\alpha^{(1+6p^n)p^{n+j}}\overline{C}_{p^n}-\alpha^{2\lambda p^{n+j}}\overline{C}_{2p^n}-\gamma^{p^j}\overline{C}_{4p^n}+\alpha^{(6+4p^n)p^{n+j}}\overline{C}_{6p^n}-\overline{C}_{8p^n}$$
$$+\alpha^{2\lambda p^{n+j}}\overline{C}_{10p^n}-\gamma^{p^j}\overline{C}_{12p^n}-\alpha^{2\lambda p^{n+j}}\overline{C}_{14p^n}-\alpha^{\mu p^{n+j}}\overline{C}_{\lambda p^n}-\alpha^{\lambda p^{n+j}}\overline{C}_{\mu p^n}-\overline{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}}$$
$$-\overline{C}_{16p^{n-j-1}}\Big\}-\phi(p^{n-j})\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\overline{C}_{4p^i}+\overline{C}_{8p^i}-\overline{C}_{16p^i}+\alpha^{2p^{i+j}}\overline{C}_{2\lambda p^i}-\alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i}-\alpha^{2p^{i+j}}\overline{C}_{2\nu p^i}\Big\}$$
$$+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{P_{i+j}\overline{C}_{p^i}-Y_{i+j}\overline{C}_{2p^i}-M_{i+j}\overline{C}_{4p^i}+V_{i+j}\overline{C}_{\lambda p^i}-R_{i+j}\overline{C}_{2\lambda p^i}+M_{i+j}\overline{C}_{4\lambda p^i}$$
$$+X_{i+j}\overline{C}_{\mu p^i}+Y_{i+j}\overline{C}_{2\mu p^i}+T_{i+j}\overline{C}_{\nu p^i}+R_{i+j}\overline{C}_{2\nu p^i}-P_{i+j}\overline{C}_{\chi p^i}-V_{i+j}\overline{C}_{\psi p^i}-X_{i+j}\overline{C}_{\xi p^i}+T_{i+j}\overline{C}_{\tau p^i}\Big]$$

$$\theta_{2p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\bar{C}_0-\alpha^{2\lambda p^{n+j}}\bar{C}_{p^n}-\gamma^{p^j}\bar{C}_{2p^n}-\bar{C}_{4p^n}+\gamma^{p^j}\bar{C}_{6p^n}+\bar{C}_{8p^n}+\alpha^{2\lambda p^{n+j}}\bar{C}_{10p^n}$$
$$-\bar{C}_{12p^n}-\gamma^{p^j}\bar{C}_{14p^n}-\beta^{p^j}\bar{C}_{\lambda p^n}+\alpha^{2\lambda p^{n+j}}\bar{C}_{\mu p^n}+\gamma^{p^j}\bar{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\bar{C}_{4p^{n-j-1}}-\bar{C}_{8p^{n-j-1}}$$
$$-\bar{C}_{16p^{n-j-1}}-\bar{C}_{4\lambda p^{n-j-1}}\Big\}-\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\bar{C}_{2p^i}+\bar{C}_{4p^i}-\bar{C}_{8p^i}-\bar{C}_{16p^i}+\alpha^{2p^{i+j}}\bar{C}_{\lambda p^i}$$
$$+\alpha^{4p^{i+j}}\bar{C}_{2\lambda p^i}+\bar{C}_{4\lambda p^i}+\alpha^{4p^{i+j}}\bar{C}_{2\mu p^i}+\alpha^{2p^{i+j}}\bar{C}_{\nu p^i}+\bar{C}_{2\nu p^i}+\alpha^{2p^{i+j}}\bar{C}_{\psi p^i}-\alpha^{4p^{i+j}}\bar{C}_{\tau p^i}\Big\}$$
$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{Y_{i+j}\bar{C}_{p^i}+M_{i+j}\bar{C}_{2p^i}+R_{i+j}\bar{C}_{\lambda p^i}+M_{i+j}\bar{C}_{2\lambda p^i}-Y_{i+j}\bar{C}_{\mu p^i}+M_{i+j}\bar{C}_{2\mu p^i}+R_{i+j}\bar{C}_{\nu p^i}$$
$$-M_{i+j}\bar{C}_{2\nu p^i}+Y_{i+j}\bar{C}_{\chi p^i}+R_{i+j}\bar{C}_{\psi p^i}-Y_{i+j}\bar{C}_{\xi p^i}-R_{i+j}\bar{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{4p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\bar{C}_0+\gamma^{p^j}\bar{C}_{p^n}-\bar{C}_{2p^n}+\bar{C}_{4p^n}-\bar{C}_{6p^n}+\bar{C}_{8p^n}-\bar{C}_{10p^n}+\bar{C}_{12p^n}-\bar{C}_{14p^n}$$
$$+\gamma^{p^j}\bar{C}_{\lambda p^n}-\gamma^{p^j}\bar{C}_{\mu p^n}+\gamma^{p^j}\bar{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\bar{C}_{2p^{n-j-1}}-\bar{C}_{4p^{n-j-1}}-\bar{C}_{8p^{n-j-1}}-\bar{C}_{16p^{n-j-1}}+\bar{C}_{2\lambda p^{n-j-1}}$$
$$-\bar{C}_{4\lambda p^{n-j-1}}+\bar{C}_{2\mu p^{n-j-1}}+\bar{C}_{2\nu p^{n-j-1}}\Big\}-\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\bar{C}_{p^i}+\bar{C}_{2p^i}-\bar{C}_{4p^i}-\bar{C}_{8p^i}-\bar{C}_{16p^i}$$
$$-\alpha^{4p^{i+j}}\bar{C}_{\lambda p^i}+\bar{C}_{2\lambda p^i}-\bar{C}_{4\lambda p^i}+\alpha^{4p^{i+j}}\bar{C}_{\mu p^i}+\bar{C}_{2\mu p^i}-\alpha^{4p^{i+j}}\bar{C}_{\nu p^i}+\bar{C}_{2\nu p^i}-\alpha^{4p^{i+j}}\bar{C}_{\psi p^i}+\alpha^{4p^{i+j}}\bar{C}_{\chi p^i}$$
$$+\alpha^{4p^{i+j}}\bar{C}_{\xi p^i}-\alpha^{4p^{i+j}}\bar{C}_{\tau p^i}\Big\}-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{M_{i+j}\bar{C}_{p^i}-M_{i+j}\bar{C}_{\lambda p^i}+M_{i+j}\bar{C}_{\mu p^i}-M_{i+j}\bar{C}_{\nu p^i}+M_{i+j}\bar{C}_{\chi p^i}$$
$$-M_{i+j}\bar{C}_{\psi p^i}+M_{i+j}\bar{C}_{\xi p^i}-M_{i+j}\bar{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{8p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\bar{C}_0-\bar{C}_{p^n}+\bar{C}_{2p^n}+\bar{C}_{4p^n}+\bar{C}_{6p^n}+\bar{C}_{8p^n}+\bar{C}_{10p^n}+\bar{C}_{12p^n}+\bar{C}_{14p^n}-\bar{C}_{\lambda p^n}-\bar{C}_{\mu p^n}$$
$$-\bar{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\bar{C}_{p^{n-j-1}}-\bar{C}_{2p^{n-j-1}}-\bar{C}_{4p^{n-j-1}}-\bar{C}_{8p^{n-j-1}}-\bar{C}_{16p^{n-j-1}}+\bar{C}_{\lambda p^{n-j-1}}-\bar{C}_{2\lambda p^{n-j-1}}-\bar{C}_{4\lambda p^{n-j-1}}+\bar{C}_{\mu p^{n-j-1}}$$
$$-\bar{C}_{2\mu p^{n-j-1}}+\bar{C}_{\nu p^{n-j-1}}-\bar{C}_{2\nu p^{n-j-1}}+\bar{C}_{\chi p^{n-j-1}}+\bar{C}_{\psi p^{n-j-1}}+\bar{C}_{\xi p^{n-j-1}}+\bar{C}_{\tau p^{n-j-1}}\Big\}-\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\bar{C}_{p^i}-\bar{C}_{2p^i}-\bar{C}_{4p^i}$$
$$-\bar{C}_{8p^i}-\bar{C}_{16p^i}+\bar{C}_{\lambda p^i}-\bar{C}_{2\lambda p^i}-\bar{C}_{4\lambda p^i}+\bar{C}_{\mu p^i}-\bar{C}_{2\mu p^i}+\bar{C}_{\nu p^i}-\bar{C}_{2\nu p^i}+\bar{C}_{\psi p^i}+\bar{C}_{\chi p^i}+\bar{C}_{\xi p^i}+\bar{C}_{\tau p^i}\Big\}$$

$$\theta_{16p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\bar{C}_0+\bar{C}_{p^n}+\bar{C}_{2p^n}+\bar{C}_{4p^n}+\bar{C}_{6p^n}+\bar{C}_{8p^n}+\bar{C}_{10p^n}+\bar{C}_{12p^n}+\bar{C}_{14p^n}+\bar{C}_{\lambda p^n}+\bar{C}_{\mu p^n}$$
$$+\bar{C}_{\nu p^n}\Big\}-p^{n-j-1}\Big\{\bar{C}_{p^{n-j-1}}+\bar{C}_{2p^{n-j-1}}+\bar{C}_{4p^{n-j-1}}+\bar{C}_{8p^{n-j-1}}+\bar{C}_{16p^{n-j-1}}+\bar{C}_{\lambda p^{n-j-1}}+\bar{C}_{2\lambda p^{n-j-1}}+\bar{C}_{4\lambda p^{n-j-1}}+\bar{C}_{\mu p^{n-j-1}}$$
$$+\bar{C}_{2\mu p^{n-j-1}}+\bar{C}_{\nu p^{n-j-1}}+\bar{C}_{2\nu p^{n-j-1}}+\bar{C}_{\chi p^{n-j-1}}+\bar{C}_{\psi p^{n-j-1}}+\bar{C}_{\xi p^{n-j-1}}+\bar{C}_{\tau p^{n-j-1}}\Big\}+\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\bar{C}_{p^i}+\bar{C}_{2p^i}+\bar{C}_{4p^i}$$
$$+\bar{C}_{8p^i}+\bar{C}_{16p^i}+\bar{C}_{\lambda p^i}+\bar{C}_{2\lambda p^i}+\bar{C}_{4\lambda p^i}+\bar{C}_{\mu p^i}+\bar{C}_{2\mu p^i}+\bar{C}_{\nu p^i}+\bar{C}_{2\nu p^i}+\bar{C}_{\psi p^i}+\bar{C}_{\chi p^i}+\bar{C}_{\xi p^i}+\bar{C}_{\tau p^i}\Big\}$$

$$
\begin{aligned}
\theta_{\lambda p^j}(x) = \frac{1}{16p^n}\Big[& \phi\left(p^{n-j}\right)\Big\{\bar{C}_0 - \alpha^{\mu p^{n+j}}\bar{C}_{p^n} - \beta^{p^j}\bar{C}_{2p^n} + \gamma^{p^j}\bar{C}_{4p^n} - \alpha^{6p^{n+j}}\bar{C}_{6p^n} - \bar{C}_{8p^n} \\
& + \beta^{p^j}\bar{C}_{10p^n} - \gamma^{p^j}\bar{C}_{12p^n} + \alpha^{6p^{n+j}}\bar{C}_{14p^n} + \alpha^{\nu p^{n+j}}\bar{C}_{\lambda p^n} + \bar{C}_{\mu p^n} - \alpha^{\lambda p^{n+j}}\bar{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\bar{C}_{8p^{n-j-1}} \\
& - \bar{C}_{16p^{n-j-1}}\Big\} - \phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\bar{C}_{2p^i} - \alpha^{4p^{i+j}}\bar{C}_{4p^i} + \bar{C}_{8p^i} - \bar{C}_{16p^i} + \alpha^{4p^{i+j}}\bar{C}_{4\lambda p^i} - \alpha^{2p^{i+j}}\bar{C}_{2\mu p^i}\Big\} \\
& + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{V_{i+j}\bar{C}_{p^i} - R_{i+j}\bar{C}_{2p^i} + M_{i+j}\bar{C}_{4p^i} - P_{i+j}\bar{C}_{\lambda p^i} - Y_{i+j}\bar{C}_{2\lambda p^i} - M_{i+j}\bar{C}_{4\lambda p^i} \\
& + T_{i+j}\bar{C}_{\mu p^i} + R_{i+j}\bar{C}_{2\mu p^i} + X_{i+j}\bar{C}_{\nu p^i} + Y_{i+j}\bar{C}_{2\nu p^i} - V_{i+j}\bar{C}_{\chi p^i} + P_{i+j}\bar{C}_{\psi p^i} - T_{i+j}\bar{C}_{\xi p^i} - X_{i+j}\bar{C}_{\tau p^i}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\theta_{2\lambda p^j}(x) = \frac{1}{16p^n}\Big[& \phi\left(p^{n-j}\right)\Big\{\bar{C}_0 - \beta^{p^j}\bar{C}_{p^n} + \gamma^{p^j}\bar{C}_{2p^n} - \bar{C}_{4p^n} - \gamma^{p^j}\bar{C}_{6p^n} + \bar{C}_{8p^n} + \gamma^{p^j}\bar{C}_{10p^n} - \bar{C}_{12p^n} - \gamma^{p^j}\bar{C}_{14p^n} \\
& - \alpha^{2\lambda p^{n+j}}\bar{C}_{\lambda p^n} + \beta^{p^j}\bar{C}_{\mu p^n} + \alpha^{2\lambda p^{n+j}}\bar{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} - \bar{C}_{16p^{n-j-1}} + \bar{C}_{4\lambda p^{n-j-1}}\Big\} \\
& - \phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\bar{C}_{p^i} - \alpha^{4p^{i+j}}\bar{C}_{2p^i} + \bar{C}_{4p^i} - \bar{C}_{8p^i} - \bar{C}_{16p^i} + \alpha^{4p^{i+j}}\bar{C}_{2\lambda p^i} + \bar{C}_{4\lambda p^i} - \alpha^{2p^{i+j}}\bar{C}_{\mu p^i} \\
& - \alpha^{4p^{i+j}}\bar{C}_{2\mu p^i} + \alpha^{4p^{i+j}}\bar{C}_{2\nu p^i} + \alpha^{2p^{i+j}}\bar{C}_{\chi p^i} - \alpha^{2p^{i+j}}\bar{C}_{\xi p^i}\Big\} - \frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{R_{i+j}\bar{C}_{p^i} - M_{i+j}\bar{C}_{2p^i} + Y_{i+j}\bar{C}_{\lambda p^i} \\
& + M_{i+j}\bar{C}_{2\lambda p^i} - R_{i+j}\bar{C}_{\mu p^i} - M_{i+j}\bar{C}_{2\mu p^i} - Y_{i+j}\bar{C}_{\nu p^i} + M_{i+j}\bar{C}_{2\nu p^i} + R_{i+j}\bar{C}_{\chi p^i} - Y_{i+j}\bar{C}_{\psi p^i} - R_{i+j}\bar{C}_{\xi p^i} - Y_{i+j}\bar{C}_{\tau p^i}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\theta_{4\lambda p^j}(x) = \frac{1}{16p^n}\Big[& \phi\left(p^{n-j}\right)\Big\{\bar{C}_0 + \gamma^{p^j}\bar{C}_{p^n} - \bar{C}_{2p^n} + \bar{C}_{4p^n} - \bar{C}_{6p^n} + \bar{C}_{8p^n} - \bar{C}_{10p^n} + \bar{C}_{12p^n} - \bar{C}_{14p^n} \\
& - \gamma^{p^j}\bar{C}_{\lambda p^n} + \gamma^{p^j}\bar{C}_{\mu p^n} - \gamma^{p^j}\bar{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\bar{C}_{2p^{n-j-1}} - \bar{C}_{4p^{n-j-1}} - \bar{C}_{8p^{n-j-1}} - \bar{C}_{16p^{n-j-1}} + \bar{C}_{2\lambda p^{n-j-1}} \\
& - \bar{C}_{4\lambda p^{n-j-1}} + \bar{C}_{2\mu p^{n-j-1}} + \bar{C}_{2\nu p^{n-j-1}}\Big\} + \phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\bar{C}_{p^i} - \bar{C}_{2p^i} + \bar{C}_{4p^i} + \bar{C}_{8p^i} + \bar{C}_{16p^i} \\
& - \alpha^{4p^{i+j}}\bar{C}_{\lambda p^i} - \bar{C}_{2\lambda p^i} + \bar{C}_{4\lambda p^i} + \alpha^{4p^{i+j}}\bar{C}_{\mu p^i} - \bar{C}_{2\mu p^i} - \alpha^{4p^{i+j}}\bar{C}_{\nu p^i} - \bar{C}_{2\nu p^i} - \alpha^{4p^{i+j}}\bar{C}_{\psi p^i} + \alpha^{4p^{i+j}}\bar{C}_{\chi p^i} \\
& + \alpha^{4p^{i+j}}\bar{C}_{\xi p^i} - \alpha^{4p^{i+j}}\bar{C}_{\tau p^i}\Big\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{M_{i+j}\bar{C}_{p^i} - M_{i+j}\bar{C}_{\lambda p^i} + M_{i+j}\bar{C}_{\mu p^i} - M_{i+j}\bar{C}_{\nu p^i} + M_{i+j}\bar{C}_{\chi p^i} \\
& - M_{i+j}\bar{C}_{\psi p^i} + M_{i+j}\bar{C}_{\xi p^i} - M_{i+j}\bar{C}_{\tau p^i}\Big\}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\theta_{\mu p^j}(x) = \frac{1}{16p^n}\Big[& \phi\left(p^{n-j}\right)\Big\{\bar{C}_0 - \alpha^{\lambda p^{n+j}}\bar{C}_{p^n} + \alpha^{2\lambda p^{n+j}}\bar{C}_{2p^n} - \gamma^{p^j}\bar{C}_{4p^n} - \alpha^{6\lambda p^{n+j}}\bar{C}_{6p^n} - \bar{C}_{8p^n} + \alpha^{2\lambda p^{n+j}}\bar{C}_{10p^n} \\
& + \gamma^{p^j}\bar{C}_{12p^n} + \alpha^{6\lambda p^{n+j}}\bar{C}_{14p^n} + \bar{C}_{\lambda p^n} + \alpha^{\nu p^{n+j}}\bar{C}_{\mu p^n} - \alpha^{\mu p^{n+j}}\bar{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\bar{C}_{8p^{n-j-1}} - \bar{C}_{16p^{n-j-1}}\Big\} \\
& - \phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\bar{C}_{4p^i} + \bar{C}_{8p^i} - \bar{C}_{16p^i} - \alpha^{2p^{i+j}}\bar{C}_{2\lambda p^i} + \bar{C}_{4\lambda p^i} + \alpha^{4p^{i+j}}\bar{C}_{2\mu p^i} + \alpha^{2p^{i+j}}\bar{C}_{\nu p^i} \\
& - \alpha^{4p^{i+j}}\bar{C}_{2\nu p^i} - \alpha^{2p^{i+j}}\bar{C}_{\psi p^i} - \alpha^{2p^{i+j}}\bar{C}_{\tau p^i}\Big\} + \frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{V_{i+j}\bar{C}_{p^i} - R_{i+j}\bar{C}_{2p^i} + M_{i+j}\bar{C}_{4p^i} - P_{i+j}\bar{C}_{\lambda p^i} \\
& - Y_{i+j}\bar{C}_{2\lambda p^i} - M_{i+j}\bar{C}_{4\lambda p^i} + T_{i+j}\bar{C}_{\mu p^i} + R_{i+j}\bar{C}_{2\mu p^i} + X_{i+j}\bar{C}_{\nu p^i} + Y_{i+j}\bar{C}_{2\nu p^i} - V_{i+j}\bar{C}_{\chi p^i} \\
& + P_{i+j}\bar{C}_{\psi p^i} - T_{i+j}\bar{C}_{\xi p^i} - X_{i+j}\bar{C}_{\tau p^i}\Big]
\end{aligned}
$$

$$\theta_{2\mu p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\overline{C}_0+\alpha^{2\lambda p^{n+j}}\overline{C}_{p^n}-\gamma^{p^j}\overline{C}_{2p^n}-\overline{C}_{4p^n}+\gamma^{p^j}\overline{C}_{6p^n}+\overline{C}_{8p^n}-\gamma^{p^j}\overline{C}_{10p^n}$$

$$-\overline{C}_{12p^n}+\gamma^{p^j}\overline{C}_{14p^n}+\beta^{p^j}\overline{C}_{\lambda p^n}-\alpha^{2\lambda p^{n+j}}\overline{C}_{\mu p^n}-\beta^{p^j}\overline{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}$$

$$-\overline{C}_{16p^{n-j-1}}+\overline{C}_{4\lambda p^{n-j-1}}\Big\}-\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\overline{C}_{2p^i}+\overline{C}_{4p^i}-\overline{C}_{8p^i}-\overline{C}_{16p^i}-\alpha^{4p^{i+j}}\overline{C}_{2\lambda p^i}$$

$$+\overline{C}_{4\lambda p^i}+\alpha^{4p^{i+j}}\overline{C}_{2\mu p^i}+\alpha^{2p^{i+j}}\overline{C}_{\nu p^i}-\alpha^{4p^{i+j}}\overline{C}_{2\nu p^i}-\alpha^{2p^{i+j}}\overline{C}_{\psi p^i}+\alpha^{2p^{i+j}}\overline{C}_{\tau p^i}\Big\}$$

$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{Y_{i+j}\overline{C}_{p^i}+M_{i+j}\overline{C}_{2p^i}+T_{i+j}\overline{C}_{\lambda p^i}-M_{i+j}\overline{C}_{2\lambda p^i}+Y_{i+j}\overline{C}_{\mu p^i}+M_{i+j}\overline{C}_{2\mu p^i}+R_{i+j}\overline{C}_{\nu p^i}$$

$$-M_{i+j}\overline{C}_{2\nu p^i}-Y_{i+j}\overline{C}_{\chi p^i}-R_{i+j}\overline{C}_{\psi p^i}-Y_{i+j}\overline{C}_{\xi p^i}+R_{i+j}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{\nu p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\overline{C}_0-\alpha^{p^{n+j}}\overline{C}_{p^n}+\beta^{p^j}\overline{C}_{2p^n}+\gamma^{p^j}\overline{C}_{4p^n}+\alpha^{6p^{n+j}}\overline{C}_{6p^n}-\overline{C}_{8p^n}-\beta^{p^j}\overline{C}_{10p^n}$$

$$-\gamma^{p^j}\overline{C}_{12p^n}-\alpha^{6p^{n+j}}\overline{C}_{14p^n}-\alpha^{\lambda p^{n+j}}\overline{C}_{\lambda p^n}-\alpha^{\mu p^{n+j}}\overline{C}_{\mu p^n}-\alpha^{\nu p^{n+j}}\overline{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}}$$

$$-\overline{C}_{16p^{n-j-1}}\Big\}+\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\overline{C}_{2p^i}-\alpha^{4p^{i+j}}\overline{C}_{4p^i}-\overline{C}_{8p^i}+\overline{C}_{16p^i}-\alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i}-\alpha^{2p^{i+j}}\overline{C}_{2\mu p^i}\Big\}$$

$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{T_{i+j}\overline{C}_{p^i}-R_{i+j}\overline{C}_{2p^i}-M_{i+j}\overline{C}_{4p^i}-X_{i+j}\overline{C}_{\lambda p^i}-Y_{i+j}\overline{C}_{2\lambda p^i}+M_{i+j}\overline{C}_{4\lambda p^i}$$

$$+V_{i+j}\overline{C}_{\mu p^i}+R_{i+j}\overline{C}_{2\mu p^i}-P_{i+j}\overline{C}_{\nu p^i}+Y_{i+j}\overline{C}_{2\nu p^i}-T_{i+j}\overline{C}_{\chi p^i}+X_{i+j}\overline{C}_{\psi p^i}+V_{i+j}\overline{C}_{\xi p^i}+P_{i+j}\overline{C}_{\tau p^i}\Big]$$

$$\theta_{2\nu p^j}(x)=\frac{1}{16p^n}\Big[\phi\left(p^{n-j}\right)\Big\{\overline{C}_0+\beta^{p^j}\overline{C}_{p^n}+\gamma^{p^j}\overline{C}_{2p^n}-\overline{C}_{4p^n}-\gamma^{p^j}\overline{C}_{6p^n}+\overline{C}_{8p^n}+\gamma^{p^j}\overline{C}_{10p^n}-\overline{C}_{12p^n}-\gamma^{p^j}\overline{C}_{14p^n}$$

$$+\alpha^{2\lambda p^{n+j}}\overline{C}_{\lambda p^n}-\beta^{p^j}\overline{C}_{\mu p^n}-\alpha^{2\lambda p^{n+j}}\overline{C}_{\nu p^n}\Big\}+p^{n-j-1}\Big\{\overline{C}_{4p^{n-j-1}}-\overline{C}_{8p^{n-j-1}}-\overline{C}_{16p^{n-j-1}}+\overline{C}_{4\lambda p^{n-j-1}}\Big\}$$

$$+\phi\left(p^{n-j}\right)\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\overline{C}_{p^i}+\alpha^{4p^{i+j}}\overline{C}_{2p^i}-\overline{C}_{4p^i}+\overline{C}_{8p^i}+\overline{C}_{16p^i}-\alpha^{4p^{i+j}}\overline{C}_{2\lambda p^i}-\overline{C}_{4\lambda p^i}-\alpha^{2p^{i+j}}\overline{C}_{\mu p^i}$$

$$+\alpha^{4p^{i+j}}\overline{C}_{2\mu p^i}-\alpha^{4p^{i+j}}\overline{C}_{2\nu p^i}+\alpha^{2p^{i+j}}\overline{C}_{\chi p^i}+\alpha^{2p^{i+j}}\overline{C}_{\xi p^i}\Big\}+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{R_{i+j}\overline{C}_{p^i}+M_{i+j}\overline{C}_{2p^i}+Y_{i+j}\overline{C}_{\lambda p^i}$$

$$-M_{i+j}\overline{C}_{2\lambda p^i}-R_{i+j}\overline{C}_{\mu p^i}+M_{i+j}\overline{C}_{2\mu p^i}-Y_{i+j}\overline{C}_{\nu p^i}-M_{i+j}\overline{C}_{2\nu p^i}+R_{i+j}\overline{C}_{\chi p^i}+Y_{i+j}\overline{C}_{\psi p^i}-R_{i+j}\overline{C}_{\xi p^i}-Y_{i+j}\overline{C}_{\tau p^i}\Big\}\Big]$$

$$\theta_{\chi p^j}(x) = \frac{1}{16p^n}\Big[\phi(p^{n-j})\Big\{\overline{C}_0 + \alpha^{\nu p^{n+j}}\overline{C}_{p^n} - \alpha^{2\nu p^{n+j}}\overline{C}_{2p^n} - \gamma^{p^j}\overline{C}_{4p^n} - \alpha^{6\nu p^{n+j}}\overline{C}_{6p^n} - \overline{C}_{8p^n}$$
$$-\alpha^{2\nu p^{n+j}}\overline{C}_{10p^n} + \gamma^{p^j}\overline{C}_{12p^n} - \alpha^{6\nu p^{n+j}}\overline{C}_{14p^n} + \alpha^{\mu p^{n+j}}\overline{C}_{\lambda p^n} + \alpha^{\lambda p^{n+j}}\overline{C}_{\mu p^n} + \overline{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}}$$
$$-\overline{C}_{16p^{n-j-1}}\Big\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{16p^i} - \alpha^{2p^{i+j}}\overline{C}_{2\lambda p^i} + \alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i} + \alpha^{2p^{i+j}}\overline{C}_{2\nu p^i}\Big\}$$
$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{P_{i+j}\overline{C}_{p^i} + Y_{i+j}\overline{C}_{2p^i} + M_{i+j}\overline{C}_{4p^i} + V_{i+j}\overline{C}_{\lambda p^i} - R_{i+j}\overline{C}_{2\lambda p^i} - M_{i+j}\overline{C}_{4\lambda p^i}$$
$$+X_{i+j}\overline{C}_{\mu p^i} - Y_{i+j}\overline{C}_{2\mu p^i} - T_{i+j}\overline{C}_{\nu p^i} - R_{i+j}\overline{C}_{2\nu p^i} - P_{i+j}\overline{C}_{\chi p^i} - V_{i+j}\overline{C}_{\psi p^i} - X_{i+j}\overline{C}_{\xi p^i} + T_{i+j}\overline{C}_{\tau p^i}\Big]$$

$$\theta_{\psi p^j}(x) = \frac{1}{16p^n}\Big[\phi(p^{n-j})\Big\{\overline{C}_0 + \alpha^{\mu p^{n+j}}\overline{C}_{p^n} - \beta^{p^j}\overline{C}_{2p^n} + \gamma^{p^j}\overline{C}_{4p^n} - \alpha^{6p^{n+j}}\overline{C}_{6p^n} - \overline{C}_{8p^n}$$
$$+\beta^{p^j}\overline{C}_{10p^n} - \gamma^{p^j}\overline{C}_{12p^n} + \alpha^{6p^{n+j}}\overline{C}_{14p^n} - \alpha^{\nu p^{n+j}}\overline{C}_{\lambda p^n} - \overline{C}_{\mu p^n} + \alpha^{\lambda p^{n+j}}\overline{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}}$$
$$-\overline{C}_{16p^{n-j-1}}\Big\} - \phi(p^{n-j})\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\overline{C}_{2p^i} - \alpha^{4p^{i+j}}\overline{C}_{4p^i} + \overline{C}_{8p^i} - \overline{C}_{16p^i} + \alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i} - \alpha^{2p^{i+j}}\overline{C}_{2\mu p^i}\Big\}$$
$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{V_{i+j}\overline{C}_{p^i} + R_{i+j}\overline{C}_{2p^i} - M_{i+j}\overline{C}_{4p^i} - P_{i+j}\overline{C}_{\lambda p^i} + Y_{i+j}\overline{C}_{2\lambda p^i} + M_{i+j}\overline{C}_{4\lambda p^i}$$
$$+T_{i+j}\overline{C}_{\mu p^i} - R_{i+j}\overline{C}_{2\mu p^i} + X_{i+j}\overline{C}_{\nu p^i} - Y_{i+j}\overline{C}_{2\nu p^i} - V_{i+j}\overline{C}_{\chi p^i} + P_{i+j}\overline{C}_{\psi p^i} - T_{i+j}\overline{C}_{\xi p^i} - X_{i+j}\overline{C}_{\tau p^i}\Big]$$

$$\theta_{\xi p^j}(x) = \frac{1}{16p^n}\Big[\phi(p^{n-j})\Big\{\overline{C}_0 + \alpha^{\lambda p^{n+j}}\overline{C}_{p^n} + \alpha^{2\lambda p^{n+j}}\overline{C}_{2p^n} - \gamma^{p^j}\overline{C}_{4p^n} + \alpha^{6\lambda p^{n+j}}\overline{C}_{6p^n} - \overline{C}_{8p^n} - \alpha^{2\lambda p^{n+j}}\overline{C}_{10p^n}$$
$$+\gamma^{p^j}\overline{C}_{12p^n} - \alpha^{6\lambda p^{n+j}}\overline{C}_{14p^n} - \overline{C}_{\lambda p^n} - \alpha^{\nu p^{n+j}}\overline{C}_{\mu p^n} + \alpha^{\mu p^{n+j}}\overline{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}} - \overline{C}_{16p^{n-j-1}}\Big\}$$
$$-\phi(p^{n-j})\sum_{i=n-j}^{n-1}\Big\{\alpha^{4p^{i+j}}\overline{C}_{4p^i} + \overline{C}_{8p^i} - \overline{C}_{16p^i} - \alpha^{2p^{i+j}}\overline{C}_{2\lambda p^i} + \alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i} + \alpha^{2p^{i+j}}\overline{C}_{2\nu p^i}\Big\}$$
$$-\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{X_{i+j}\overline{C}_{p^i} - Y_{i+j}\overline{C}_{2p^i} + M_{i+j}\overline{C}_{4p^i} - T_{i+j}\overline{C}_{\lambda p^i} - R_{i+j}\overline{C}_{2\lambda p^i} + M_{i+j}\overline{C}_{4\lambda p^i} - P_{i+j}\overline{C}_{\mu p^i}$$
$$+Y_{i+j}\overline{C}_{2\mu p^i} + V_{i+j}\overline{C}_{\nu p^i} + R_{i+j}\overline{C}_{2\nu p^i} - X_{i+j}\overline{C}_{\chi p^i} - T_{i+j}\overline{C}_{\psi p^i} + P_{i+j}\overline{C}_{\xi p^i} - V_{i+j}\overline{C}_{\tau p^i}\Big]$$

$$\theta_{\tau p^j}(x) = \frac{1}{16p^n}\Big[\phi(p^{n-j})\Big\{\overline{C}_0 + \alpha^{p^{n+j}}\overline{C}_{p^n} + \beta^{p^j}\overline{C}_{2p^n} + \gamma^{p^j}\overline{C}_{4p^n} + \alpha^{6p^{n+j}}\overline{C}_{6p^n} - \overline{C}_{8p^n} - \beta^{p^j}\overline{C}_{10p^n}$$
$$-\gamma^{p^j}\overline{C}_{12p^n} - \alpha^{6p^{n+j}}\overline{C}_{14p^n} + \alpha^{\lambda p^{n+j}}\overline{C}_{\lambda p^n} + \alpha^{\mu p^{n+j}}\overline{C}_{\mu p^n} + \alpha^{\nu p^{n+j}}\overline{C}_{\nu p^n}\Big\} + p^{n-j-1}\Big\{\overline{C}_{8p^{n-j-1}}$$
$$-\overline{C}_{16p^{n-j-1}}\Big\} + \phi(p^{n-j})\sum_{i=n-j}^{n-1}\Big\{\alpha^{2p^{i+j}}\overline{C}_{2p^i} + \alpha^{4p^{i+j}}\overline{C}_{4p^i} - \overline{C}_{8p^i} + \overline{C}_{16p^i} - \alpha^{4p^{i+j}}\overline{C}_{4\lambda p^i} - \alpha^{2p^{i+j}}\overline{C}_{2\mu p^i}\Big\}$$
$$+\frac{1}{p^j}\sum_{i=0}^{n-j-1}\Big\{T_{i+j}\overline{C}_{p^i} + R_{i+j}\overline{C}_{2p^i} + M_{i+j}\overline{C}_{4p^i} - X_{i+j}\overline{C}_{\lambda p^i} + Y_{i+j}\overline{C}_{2\lambda p^i} - M_{i+j}\overline{C}_{4\lambda p^i}$$
$$-V_{i+j}\overline{C}_{\mu p^i} - R_{i+j}\overline{C}_{2\mu p^i} - P_{i+j}\overline{C}_{\nu p^i} - Y_{i+j}\overline{C}_{2\nu p^i} - T_{i+j}\overline{C}_{\chi p^i} + X_{i+j}\overline{C}_{\psi p^i} + V_{i+j}\overline{C}_{\xi p^i} + P_{i+j}\overline{C}_{\tau p^i}\Big]$$

where β is 2nd root of unity and γ is 4th root of unity in some extension field of F. Further, $T_j, P_j, V_j, R_j, M_j, Y_j$ can be obtained by using the relations $\theta_{sp^j}\left(\alpha^{sp^j}\right) = 1$ and $\theta_{sp^j}\left(\alpha^{tp^j}\right) = 0,\ s \neq t$.

Proof. By definition,

$$\theta_s(x) = \frac{1}{16p^n}\Big[\varepsilon_0^s\overline{C}_0 + \varepsilon_{p^n}^s\overline{C}_{p^n} + \varepsilon_{2p^n}^s\overline{C}_{2p^n} + \varepsilon_{4p^n}^s\overline{C}_{4p^n} + \varepsilon_{6p^n}^s\overline{C}_{6p^n} + \varepsilon_{8p^n}^s\overline{C}_{8p^n} + \varepsilon_{10p^n}^s\overline{C}_{10p^n}$$

$$+\varepsilon_{12p^n}^s\overline{C}_{12p^n} + \varepsilon_{14p^n}^s\overline{C}_{14p^n} + \varepsilon_{\lambda p^n}^s\overline{C}_{\lambda p^n} + \varepsilon_{\mu p^n}^s\overline{C}_{\mu p^n} + \varepsilon_{\nu p^n}^s\overline{C}_{\nu p^n} + \sum_{i=0}^{n-1}\Big\{\varepsilon_{p^i}^s\overline{C}_{p^i} + \varepsilon_{2p^i}^s\overline{C}_{2p^i} + \varepsilon_{4p^i}^s\overline{C}_{4p^i}$$

$$+\varepsilon_{8p^i}^s\overline{C}_{8p^i} + \varepsilon_{16p^i}^s\overline{C}_{16p^i} + \varepsilon_{\lambda p^i}^s\overline{C}_{\lambda p^i} + \varepsilon_{2\lambda p^i}^s\overline{C}_{2\lambda p^i} + \varepsilon_{4\lambda p^i}^s\overline{C}_{4\lambda p^i} + \varepsilon_{\mu p^i}^s\overline{C}_{\mu p^i} + \varepsilon_{2\mu p^i}^s\overline{C}_{2\mu p^i} + \varepsilon_{\nu p^i}^s\overline{C}_{\nu p^i}$$

$$+\varepsilon_{2\nu p^i}^s\overline{C}_{2\nu p^i} + \varepsilon_{\chi p^i}^s\overline{C}_{\chi p^i} + \varepsilon_{\psi p^i}^s\overline{C}_{\psi p^i} + \varepsilon_{\xi p^i}^s\overline{C}_{\xi p^i} + \varepsilon_{\tau p^i}^s\overline{C}_{\tau p^i}\Big\}\Big]$$

For $s = 0$, $\varepsilon_k^0 = \sum_{s\in\Omega_0}\alpha^s = \alpha^{k0} = 1$ and hence expression for $\theta_0(x)$ is obtained.

Remaining expressions can be obtained by using lemma 3.5 – 3.8 and lemma 3.10 -3.18.

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