

Generalized compound Rayleigh model: A Bayes study using MCMC approach

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Abstract

In this paper, an attempt has been made to develop Bayesian inferential procedure for the generalized compound Rayleigh model. The parameters are estimated using likelihood based inferential procedure: classical as well as Bayesian. The maximum likelihood estimates and associated probability intervals are obtained. The Bayesian estimates of the parameters of generalized compound Rayleigh distribution are obtained using Markov chain Monte Carlo (MCMC) simulation method. The Bayes estimates of parametric functions such as hazard and reliability functions are also discussed. The probability intervals for parameters, hazard and reliability functions are computed. The posterior predictive check procedure is used for evaluating model fit. All the Bayesian computations are performed in Open BUGS and R software. A real data set is analyzed for illustration of the proposed Bayesian approach.

Key Words: Generalized compound Rayleigh distribution, Markov chain Monte Carlo, Bayesian estimation, Open BUGS.

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1. INTRODUCTION

There are situations where the hazard rate function of the underlying survival model is non-monotone, in particular unimodal or inverted bathtub hazard rate. Hence, classical survival models, such as exponential, Weibull and gamma, are not suitable in these situations. Rayleigh distribution has a linearly increasing failure rate. Thus the hazard function is the motivation for the specific family of models. Generalising and compounding of the probability density functions form the basis for the development of new models, Marshall and Olkin (2007). Greenwich (1992) introduced a parametric two-parameter survival model with unimodal hazard rate function which is compound (or gamma-mixed) Rayleigh model, Ghitany

(2001). Mostert *et al.* (1999) also considered a gamma mixtures of the Rayleigh distribution and obtained the compound Rayleigh model with a unimodal hazard function. Bekker *et al.* (2000) introduced an additional shape parameter to compound Rayleigh distribution and named it as the generalized compound Rayleigh (GCR) model. Bain and Engelhardt (1991) studied this distribution (also known as the compound Weibull distribution Dubey, (1968)) from a Poisson process perspective. The flexible failure rate makes generalized compound Rayleigh model attractive for modelling lifetimes of patients showing characteristics of a random hazard rate. Al-Hossain (2013) obtained the inferential procedures for compound Rayleigh parameters with progressively type-II censored data. The Bayes estimation of generalized compound Rayleigh distribution with change point is discussed by Pandya *et al.* (2014). It may be noted that most of the cited literature is confined to classical inferential procedures. The Bayesian methods are discussed for mathematically tractable priors. The aim of this paper to apply the Bayesian method using Markov chain Monte Carlo (MCMC) simulation approach. A major difficulty towards the implementation of Bayesian procedure is that of obtaining the posterior distribution. The process often requires the integration, which is very difficult to calculate not only for high-dimensional

complex models even if dealing with low-dimensional models. In such a situation, Monte Carlo Markov chain (MCMC) methods are useful to simulate the deviates from the posterior density to obtain the various estimates etc. The initiation of MCMC method has flourished Bayesian statistics. The freely available software package known as Bayesian inference using Gibbs sampling (BUGS) has been in the forefront of this explosion since the mid-1990s. However, more recent advances in this software, leading first to WinBUGS and now to an open-source version Open BUGS, Thomas *et al.* (2006), Thomas (2010) and Lunn *et al.* (2013), including interfaces to the open-source statistical package R, (R Development Core Team, 2016), have brought MCMC to a wider audience. We shall make use of OpenBUGS and R software in our present study. For Bayesian analysis, we need to assume a prior distribution for the model parameters. In this paper, Bayesian analysis has been performed under different loss functions assuming independent priors for the parameters. The rest of this paper is organized as follows. In Section 2, we describe the generalized compound Rayleigh model and its properties. The maximum likelihood estimation of the parameters and associated confidence intervals are given in Section 3. In Section 4, the Bayesian model formulation is discussed under independent priors for the parameters, gamma priors for shape parameters and uniform prior for scale parameter. The data for illustration of methodology is given in Section 5. In this section, the ML estimators of the parameters, approximate confidence intervals are presented. We cover Bayesian analysis using the MCMC simulation in Section 6. In this section, the Bayes estimates and credible intervals of parameters, hazard and reliability functions are presented. In Section 7 we consider the predictive check method in order to give an assessment of the performance of the model for the given data. Section 8 concludes the paper.

2. THE MODEL

Assume the Weibull distribution with hazard function parameterized in terms of scale θ and shape λ :

$$h(x) = \lambda\theta x^{\lambda-1} \quad ; x > 0, \theta > 0, \lambda > 0$$

to model the lifetimes of cancer patients. The corresponding probability density function is

$$f(x|\theta) = \lambda\theta x^{\lambda-1} \exp(-\theta x^\lambda).$$

In a population of cancer patients there may be variation in the hazard function because of unique differences between patients. Assuming the same shape parameter λ , this variation can be described and characterized by the parameter θ . Now, assigning a gamma distribution with

shape parameter α and scale parameter β on θ . The pdf of θ is

$$g(\theta) = \frac{(1/\beta)^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\theta/\beta) \quad ; \theta > 0, \alpha > 0, \beta > 0.$$

Hence, the lifetime of a patient selected at random from such a mixed population is a compound Weibull distribution (Dubey, 1968) with probability density function

$$f(x) = \int_0^\infty f(x|\theta)g(\theta)d\theta = \alpha\beta\lambda x^{\lambda-1} (1 + \beta x^\lambda)^{-(\alpha+1)}$$

Bekker *et al.* (2000) obtained the same by adding a shape parameter to compound Rayleigh distribution and named it as the generalized compound Rayleigh(GCR) model. Thus we will refer it as GCR model. The cumulative distribution function(cdf) of the generalized compound Rayleigh distribution with three parameters is given by,

$$F(x; \alpha, \beta, \lambda) = 1 - (1 + \beta x^\lambda)^{-\alpha} \quad ; x > 0. \tag{1}$$

where $\alpha > 0$ and $\lambda > 0$ are shape parameters and $\beta > 0$ is a scale parameter. The generalized compound Rayleigh model includes Beta-Prime, Lomax, Pareto second kind, Burr XII and Compound Rayleigh well-known distributions. The generalized compound Rayleigh distribution will be denoted by $GCR(\alpha, \beta, \lambda)$. The probability density function (pdf) is given by

$$f(x; \alpha, \beta, \lambda) = \alpha\beta\lambda x^{\lambda-1} (1 + \beta x^\lambda)^{-(\alpha+1)} \quad ; x > 0. \tag{2}$$

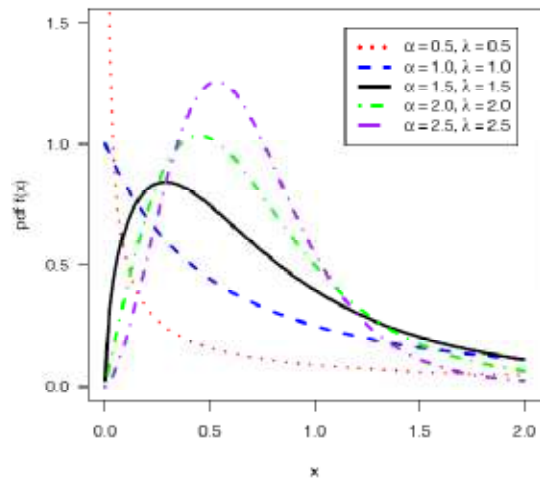


Figure 1: The probability density functions of GCR distribution for different values of α and λ , for $\beta = 1$

The reliability/survival function is

$$R(x; \alpha, \beta, \lambda) = (1 + \beta x^\lambda)^{-\alpha} \quad ; x > 0 \tag{3}$$

The hazard rate function is

$$h(x; \alpha, \beta, \lambda) = \frac{\alpha\beta\lambda x^{\lambda-1}}{(1 + \beta x^\lambda)} \quad (4)$$

Some of the typical generalized compound Rayleigh density functions for different values of α and λ for $\beta = 1$ are depicted in Figure 1. It is clear from the Figure 1 that the density function of the generalized compound Rayleigh distribution can take different shapes.

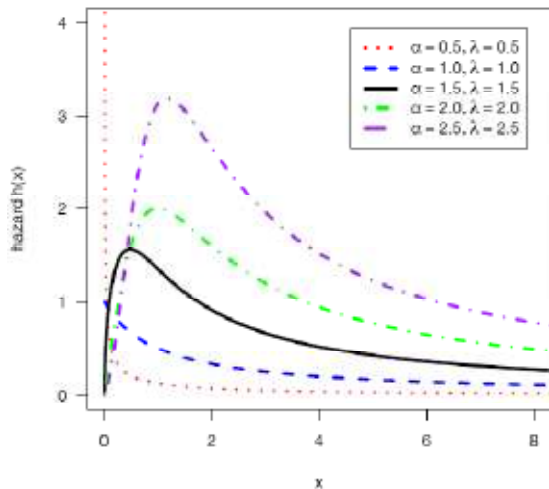


Figure 2: The hazard function of GCR distribution for different values of α and λ , for $\beta = 1$

The hazard function of GCR distribution exhibits flexible behaviour and includes unimodal failure rate, Figure 2. Greenwich (1992) mentioned that the hazard function of the Weibull distribution cannot represent a failure rate that has a relatively high rate of failure in the middle of the expected lifetime. For $\lambda \geq 1$ this hazard function has one critical point at $x = \{(\lambda - 1)/\beta\}^{1/\lambda}$. For λ fixed the hazard function shifts to the right for decreasing values of β , i.e. it may include an increasing hazard function for most of the lifetimes. The quantile function is given by

$$x_p = \left[\frac{1}{\beta} \left\{ (1-p)^{-1/\alpha} - 1 \right\} \right]^{1/\lambda}; 0 < p < 1. \quad (5)$$

The random deviate can be generated by

$$x = \left[\frac{1}{\beta} \left\{ (1-u)^{-1/\alpha} - 1 \right\} \right]^{1/\lambda}; 0 < u < 1. \quad (6)$$

where u has the $U(0, 1)$ distribution.

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, the maximum likelihood estimators (MLE's) of the parameters of $GCR(\alpha, \beta, \lambda)$ distribution are obtained and their asymptotic properties are used to obtain approximate confidence intervals based on MLE's.

Let $\underline{x} = (x_1, \dots, x_n)$ be a random sample of size n from $GCR(\alpha, \beta, \lambda)$, then the log-likelihood function $\ell(\alpha, \beta, \lambda | \underline{x})$ can be written as;

$$\ell(\alpha, \beta, \lambda | \underline{x}) = n \log \alpha + n \log \beta + n \log \lambda + (\lambda - 1) \sum_{i=1}^n \log x_i - (\alpha + 1) \sum_{i=1}^n \log(1 + \beta x_i^\lambda) \quad (7)$$

To obtain the MLE's of α, β and λ , we can maximize (7) directly with respect to α, β and λ or we can solve the following three non-linear equations using Newton-Raphson method.

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \log(1 + \beta x_i^\lambda) = 0$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - (\alpha + 1) \sum_{i=1}^n \frac{x_i^\lambda}{(1 + \beta x_i^\lambda)} = 0$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \log x_i - \beta(\alpha + 1) \sum_{i=1}^n \frac{x_i^\lambda \log x_i}{(1 + \beta x_i^\lambda)} = 0$$

The $100(1 - \gamma)\%$ confidence intervals for α, β and λ can be obtained from the usual asymptotic normality of the maximum likelihood estimators with $var(\hat{\alpha}), var(\hat{\beta})$

and $var(\hat{\lambda})$ estimated from the inverse of the observed Fisher information matrix, that is, the inverse of the matrix of second derivatives of the log-likelihood function locally at $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$. Hence, from the asymptotic normality of MLEs, approximate $100(1 - \gamma)\%$ confidence intervals for α, β and λ can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2} \sqrt{var(\hat{\alpha})}; \hat{\beta} \pm z_{\gamma/2} \sqrt{var(\hat{\beta})}; \hat{\lambda} \pm z_{\gamma/2} \sqrt{var(\hat{\lambda})}$$

where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

4. BAYESIAN MODEL FORMULATION

Prior and Posterior distributions

The Bayesian model is constructed by specifying the prior distributions for the model parameters α, β and λ .

We assume the independent gamma prior for $\alpha \sim G(a_1, b_1)$, $\lambda \sim G(a_3, b_3)$ and uniform priors $\beta \sim U(a_2, b_2)$ as

$$p(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} \exp(-b_1 \alpha) ; \alpha > 0, (a_1, b_1) > 0$$

$$p(\beta) = \frac{1}{b_2 - a_2} ; a_2 < \beta < b_2.$$

and

$$p(\lambda) = \frac{b_3^{a_3}}{\Gamma(a_3)} \lambda^{a_3-1} \exp(-b_3 \lambda) \quad ; \lambda > 0, (a_3, b_3) > 0$$

Given a set of data $\underline{x} = (x_1, \dots, x_n)$ from $GCR(\alpha, \beta, \lambda)$, the likelihood function is given by

$$L(\alpha, \beta, \lambda | \underline{x}) = (\alpha\beta\lambda)^n \prod_{i=1}^n x_i^{\lambda-1} (1 + \beta x_i^\lambda)^{-(\alpha+1)} \quad (8)$$

Combining the likelihood function, $L(\alpha, \beta, \lambda | \underline{x})$ from (8) and the priors of α , β and λ through Bayes theorem, we get the joint posterior distribution up to proportionality for α , β and λ as follows:

$$p(\alpha, \beta, \lambda | \underline{x}) \propto \alpha^{a_1+n-1} \beta^n \lambda^{a_3+n-1} \left(\prod_{i=1}^n x_i^{\lambda-1} \right) \exp(-b_1\alpha - b_3\lambda) T_1 \quad (9)$$

$$\text{where } T_1 = \prod_{i=1}^n (1 + \beta x_i^\lambda)^{-(\alpha+1)} \quad (10)$$

The posterior is complicated and no close form inferences are possible. Therefore, we propose to consider MCMC methods to simulate samples from the posterior so that sample-based inferences can be drawn. The samples are generated by running a suitably constructed Markov chain that converges to the target distribution which, in the present case, is the posterior distribution $p(\alpha, \beta, \lambda | \underline{x})$. These chains can be constructed using different methods, but all of them, including the Gibbs sampler (Gelfand and Smith, 1990), are special cases of the general framework of Metropolis *et al.* (1953) and Hastings (1970).

Gibbs Sampler : Algorithm

The full conditionals of α , β and λ , upto proportionality to implement the Gibbs sampler, can be specified as

1. Full conditional of α given β , λ and \underline{x} :

$$p(\alpha | \beta, \lambda, \underline{x}) \propto \alpha^{a_1+n-1} \exp(-b_1\alpha) T_1 \quad (11)$$

2. Full conditional of β given α , λ and \underline{x}

$$p(\beta | \alpha, \lambda, \underline{x}) \propto \beta^n T_1 \quad (12)$$

3. Full conditional of λ given α , β and \underline{x} :

$$p(\lambda | \alpha, \beta, \underline{x}) \propto \lambda^{a_3+n-1} \exp(-b_3\lambda) \left(\prod_{i=1}^n x_i^{\lambda-1} \right) T_1 \quad (13)$$

As the three-parameter generalized compound Rayleigh distribution is not available in OpenBUGS, it requires incorporation of a module in *ReliaBUGS* which is subsystem of OpenBUGS. A module *dgen.cpd.rayleigh(alpha, beta, lambda)* is written in Component Pascal for three-parameter generalized compound Rayleigh, the corresponding computer program can be obtained from authors, to perform full Bayesian analysis in OpenBUGS using the method

described in Thomas *et al.* (2006), Thomas (2010), Kumar *et al.* (2010) and Lunn *et al.* (2013).

Gibbs Sampler: Implementation

1. Select an initial value $\underline{\theta}^{(0)} = (\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)})$ to start the chain.
2. Suppose at the i^{th} -step, $\underline{\theta} = (\alpha, \beta, \lambda)$ takes the value $\underline{\theta}^{(i)} = (\alpha^{(i)}, \beta^{(i)}, \lambda^{(i)})$ then from full conditionals, we generate $\alpha^{(i+1)}$ from $p(\alpha | \beta^{(i)}, \lambda^{(i)}, \underline{x})$
 $\beta^{(i+1)}$ from $p(\beta | \alpha^{(i+1)}, \lambda^{(i)}, \underline{x})$ and
 $\lambda^{(i+1)}$ from $p(\lambda | \alpha^{(i+1)}, \beta^{(i+1)}, \underline{x})$
3. This completes a transition from $\underline{\theta}^{(i)}$ to $\underline{\theta}^{(i+1)}$
4. Repeat Step 2, N times and obtain $(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)})$; $j = 1, 2, \dots, N$.

Abandoning first samples N_0 as ‘burn in’, remaining $N - N_0$ samples are used to obtain Bayesian estimators (Posterior means) and posterior variances of (α, β, λ) with respect to the SEL function as

$$\hat{\theta}_B = \frac{1}{N - N_0} \sum_{i=N_0+1}^N \theta_i ; \quad var(\hat{\theta}_B) = \frac{1}{N - N_0} \sum_{i=N_0+1}^N (\theta_i - \hat{\theta}_B)^2 ,$$

where θ can be α , β and λ .

5. DATA

For illustrative purposes of the proposed methodology, we performed a real data analysis. The real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients, Lee and Wang (2003):

- 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

Computation of MLE and Model Validation

The estimation of the parameter of proposed model is obtained by the method of maximum likelihood (ML) estimation. The maximum likelihood estimates (MLEs)

are obtained by direct maximization of the log-likelihood function $\ell(\alpha, \beta, \lambda)$ given in (7). The Table 1 shows the ML estimates, standard error (se) and 95 % confidence intervals for parameters α, β and λ . The maximized value of loglikelihood is $\ell(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = -409.7399$. In figure 3, we plot the profiles of the negative log-likelihood function of α, β and λ for the given data. From fig. 5 we show that the likelihood equations have a unique solution. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) can be used to determine which model is most appropriate for the given data. For the given set AIC=825.480 and BIC= 834.036.

Table 1: MLE, standard error and 95% confidence interval (CI)

Parameter	MLE	Std. Error	95% CI
alpha	2.07012	0.95742	(0.1936, 3.9467)
beta	0.02868	0.01136	(0.0064, 0.0509)
lambda	1.4276	0.17675	(1.0812, 1.7740)

The advantage of this procedure is that it runs immediately using existing statistical packages such as R software (R Development Core Team, 2016). We consider the quasi Newton-Raphson algorithm in R (Rizzo, 2008) to compute the MLs.

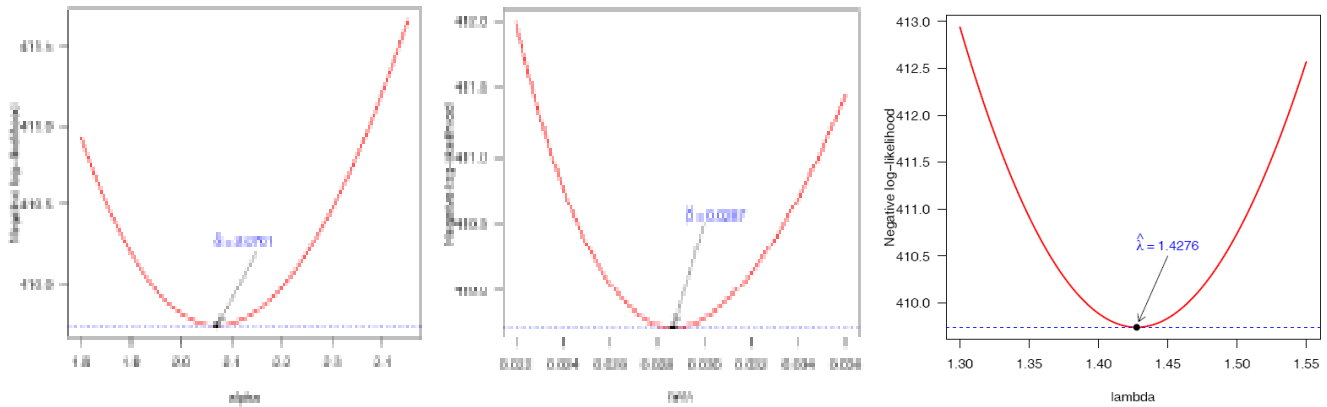


Figure 3: The profile of negative log-likelihood functions of α, β and λ

To check the validity of the model we compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by method of maximum likelihood is 0.0351 and the corresponding p -value is 0.9975.

The high p -value suggests that fit is satisfactory. We have plotted the empirical distribution function and the fitted distribution function in Figure 4. The P-P and Q-Q plots for the fitted model are shown in Figure 5 and Figure 6. It can be seen that the fitted generalized compound Rayleigh distribution provides good fit to the given data.

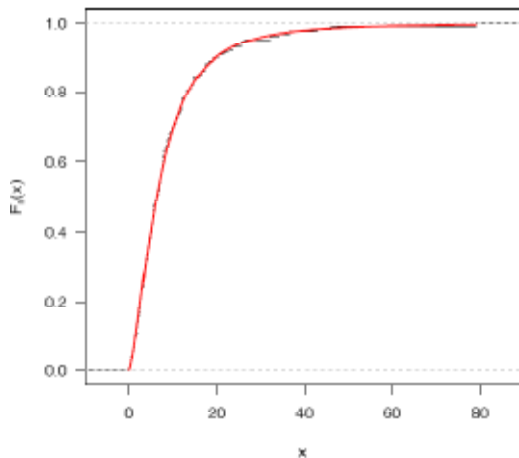


Figure 4: The graph of empirical and fitted distribution function

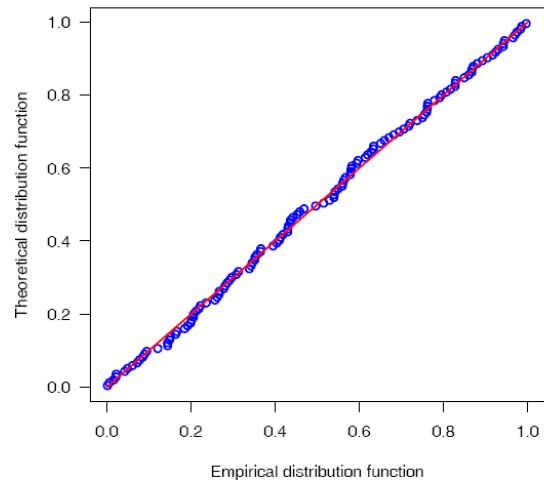


Figure 5: Probability-Probability (P-P) plot using MLEs as estimate

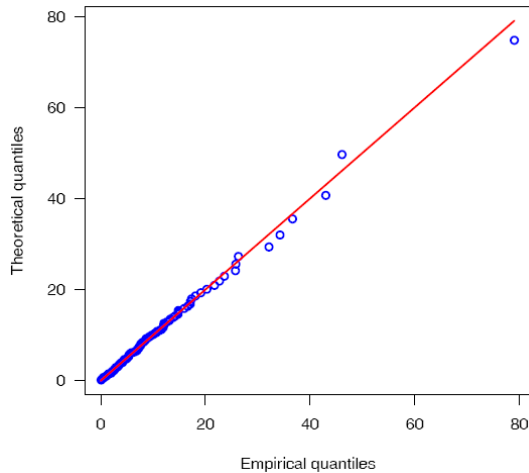


Figure 6: Quantile-Quantile(Q-Q) plot using MLEs as estimate

6. BAYESIAN ANALYSIS

OpenBUGS script for the Bayesian analysis model

```

{
  for(i in 1 : N)
  {
    x[i] ~ dgen.cpd.rayleigh(alpha, beta, lambda)
    f[i] <- density(x[i], x[i])
    reliability[i] <- R(x[i], x[i])
  }
# Prior distributions of the model parameters
alpha ~ dgamma(0.001, 0.001)
beta ~ dunif(0.0, 5.0)
lambda ~ dgamma(0.001, 0.001)
}
Data
  list(N=128, c(0.08,...,79.05))
Initial values
list(alpha=1.0, beta= 0.1, lambda=1.0) # Chain 1
list(alpha=5.0, beta=0.5, lambda=5.0) # Chain 2
    
```

We assume the independent gamma priors for $\alpha \sim G(a_1, b_1)$, $\lambda \sim G(a_3, b_3)$ and uniform prior for $\beta \sim U(a_2, b_2)$ with hyper-parameter values $a_1 = 0.001$, $b_1 = 0.001$, $a_2 = 0.0$, $b_2 = 5.0$, $a_3 = 0.001$ and $b_3 = 0.001$. We run the model to generate two Markov chains at the length of 30,000 with different starting points of the parameters. We have chosen initial values $(\alpha = 1.0, \beta = 0.1, \lambda = 1.0)$ for the first chain and $(\alpha = 5.0, \beta = 0.5, \lambda = 5.0)$ for the second chain. The convergence is monitored using trace and ergodic mean plots. We find that the Markov Chains converge together after approximately 1000 observations. Therefore, burnin of 5000 samples is more than enough to erase the effect of starting point(initial values). Finally, samples of size

5000 are formed from the posterior by picking up equally spaced every fifth outcome (to minimize the auto correlation among the generated deviates.), i.e. thin=5, starting from 5001. Therefore, we have the posterior sample $(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}) ; j = 1, \dots, 5000$ from chain 1 and $(\alpha_2^{(j)}, \beta_2^{(j)}, \lambda_2^{(j)}) ; j = 1, \dots, 5000$ from chain 2. We have considered the chain 1 for analysis as well as for convergence diagnostic plots, the chain 2 produces the similar result.

Convergence diagnostics

It is well known that MCMC analysis provides reliable results only when the chains have run sufficiently large number of times and reached to the stationary distribution. In the existing literature of MCMC, a number of tools to assess the convergence of chains. The important graphical diagnostics are the trace and the ergodic mean plots. Figure 7(left panel) shows the sequential realizations of the parameters of the model. The ergodic mean is computed as the mean of all sampled values up to and including that at a given iteration. The convergence pattern based on ergodic averages is shown in Figure 7 (right panel).

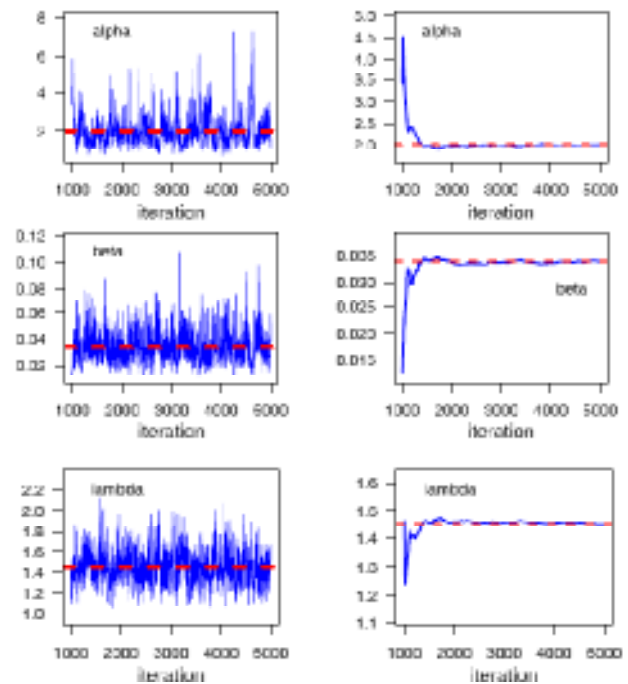


Figure 7: The trace plot (left panel) and the ergodic mean plot(right panel) for alpha, beta and lambda.

The above Figures are drawn by using R software, is enough to show that the chains in the present analysis have converged.

Posterior analysis

(a) Numerical summary

The numerical summary is presented for $(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}) ; j = 1, \dots, 5000$ from chain 1. We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for generalized compound Rayleigh distribution. The MCMC results of the posterior mean, mode, standard deviation, first quartile, median, third quartile, 2.5th percentile, 97.5th percentile, skewness, kurtosis of the parameters α, β and λ are displayed in Table 2. The Bayes estimates under absolute and zero-one loss functions are posterior median and mode, respectively.

Table 2: Numerical summaries based on MCMC sample of posterior characteristics

Characteristics	alpha	beta	lambda
Mean	2.1883	0.0329	1.4378
Standard Deviation	1.0568	0.0129	0.1784
2.5th Percentile(P _{2.5})	0.9176	0.0127	1.1310
First Quartile (Q ₁)	1.4360	0.0236	1.3090
Median	1.9105	0.0312	1.4250
Third Quartile (Q ₃)	2.6460	0.0405	1.5480
97.5th Percentile(P _{97.5})	5.0306	0.0611	1.8260
Mode	1.5402	0.0276	1.4077
Skewness	1.3891	0.7851	0.4602
Kurtosis	2.0531	1.0653	0.2138

Highest probability density (HPD): A special type of Bayesian interval is the *highest posterior density* (HPD) interval. It is built so as to include the values of the parameters that have the highest posterior probability (the most likely values). When the posterior is symmetric and has a single peak (is unimodal), credible and HPD intervals coincide. With very skewed posterior distributions, however, the two intervals look very different. The highest probability density (HPD) credible intervals for α and λ are constructed by using Chen and Shao (1999) algorithm under the assumption of unimodal marginal posterior distribution. Let $(\alpha_{(j)}; j = 1, 2, \dots, M)$ be the corresponding ordered MCMC sample of $(\alpha^{(j)}; j = 1, 2, \dots, M)$. Then, the $100(1-\gamma)\%$ HPD intervals for α is $(\alpha_{(k^*)}, \alpha_{(k^* + [(1-\gamma)M])})$, where k^* is chosen so that

$$\alpha_{(k^* + [(1-\gamma)M])} - \alpha_{(k^*)} = \min_{1 \leq k \leq M - [(1-\gamma)M]} (\alpha_{(k + [(1-\gamma)M])} - \alpha_{(k)})$$

Here $[\chi]$ denotes the largest integer less than or equal to χ . In the same fashion, one can also obtain the Bayes

HPD credible intervals for β and λ . Table 3 shows the HPD and credible intervals for parameters alpha, beta and lambda.

Table 3: HPD and Credible intervals

Parameter	Credible Interval	HPD Credible Interval
alpha	(0.9176, 5.0306)	(0.7304, 4.414)
beta	(0.0127, 0.0611)	(0.0111, 0.0586)
lambda	(1.131, 1.826)	(1.102, 1.791)

(b) Visual summary

The visual graphs include the boxplot, density strip plot, histogram, marginal posterior density estimate and rug plots for the parameters. We have also superimposed the 95% HPD intervals.

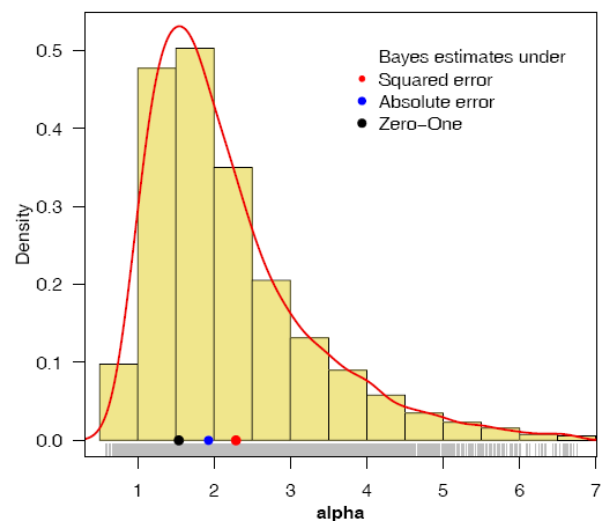


Figure 8a: Histogram and marginal posterior density

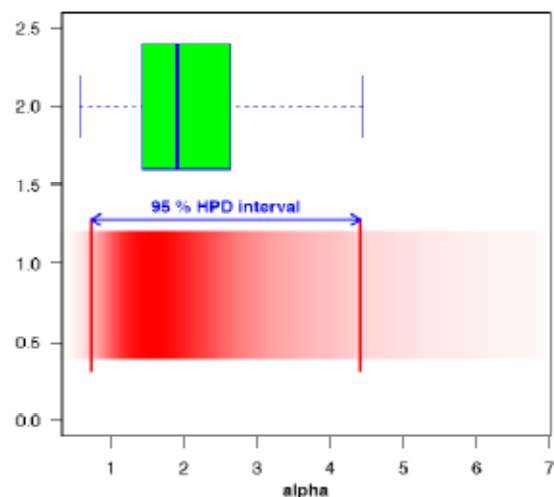


Figure 8(b): Boxplot and density strip of α , based on posterior sample.

The density strip plot introduced by Jackson (2008) for a univariate distribution as a shaded rectangular strip, whose darkness at a point is proportional to the probability density.

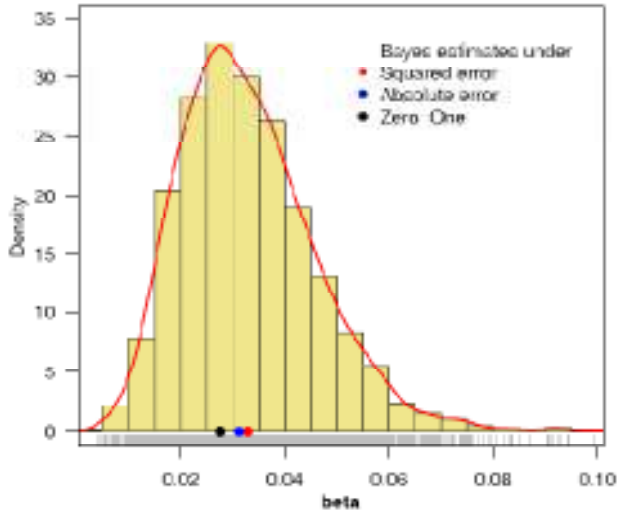


Figure 9(a): Histogram and marginal posterior density

These graphs provide almost complete picture of the posterior uncertainty about the parameters. We have used the posterior sample $(\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)})$; $j = 1, \dots, 5000$ to draw these graphs.

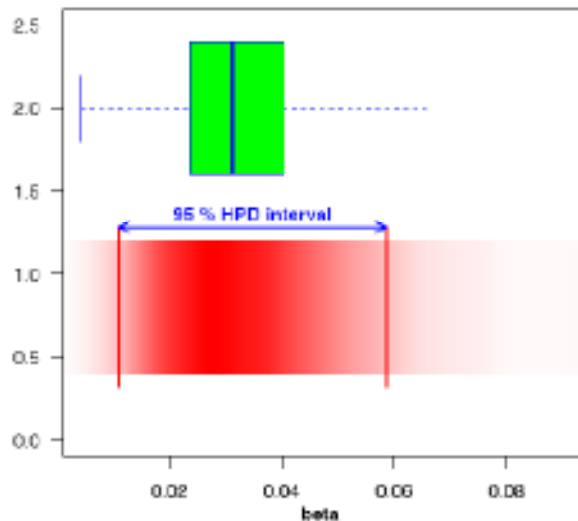


Figure 9(b): Boxplot and density strip of beta

The kernel density estimates have been drawn using R software with the assumption of Gaussian kernel and properly chosen values of the bandwidths. It can be seen that λ is symmetric whereas α and β show positive skewness.

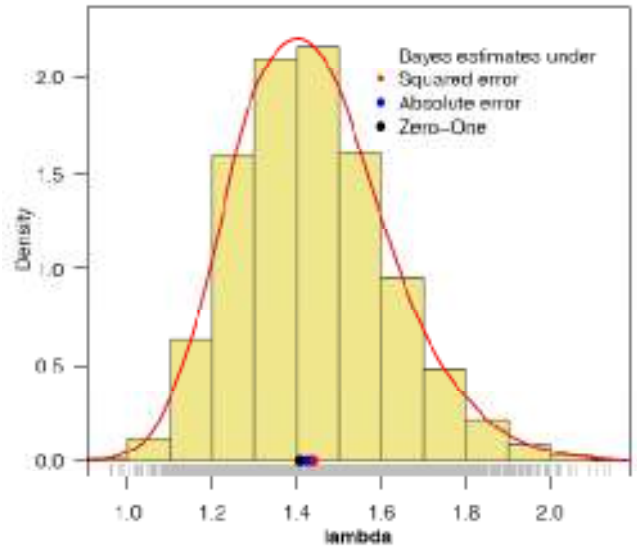


Figure 10(a): Histogram, marginal posterior density and 95% HPD interval based on posterior sample

Figure 8(a) represents the histogram, marginal posterior density for α . The boxplot, 95% HPD interval and the density strip plot are displayed in Figure 8(b).

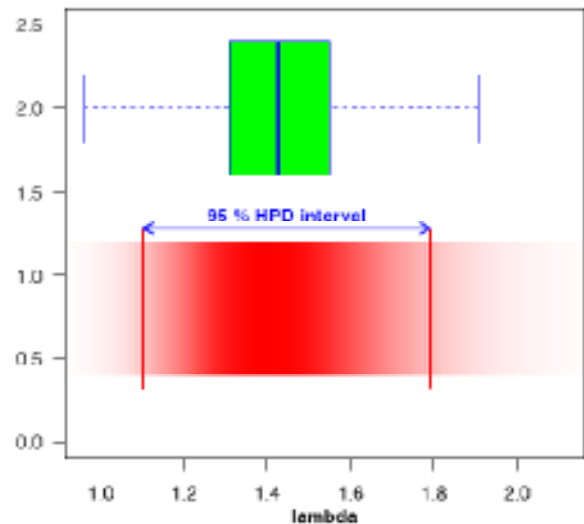


Figure 10(b): Boxplot and density strip of lambda

We have plotted the similar graphs in Figure 9(a) and (b) for β . The plots for λ are shown in Figure 10(a) and Figure 10(b).

Comparison with MLE

In Figure 11 the density functions $f(x; \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ using MLEs and Bayesian estimates, computed via MCMC samples under gamma priors for α and λ and uniform prior for β , are plotted. We observe in the Figure 11, the

MLEs and the Bayes estimates are quite close and fit the data very well.

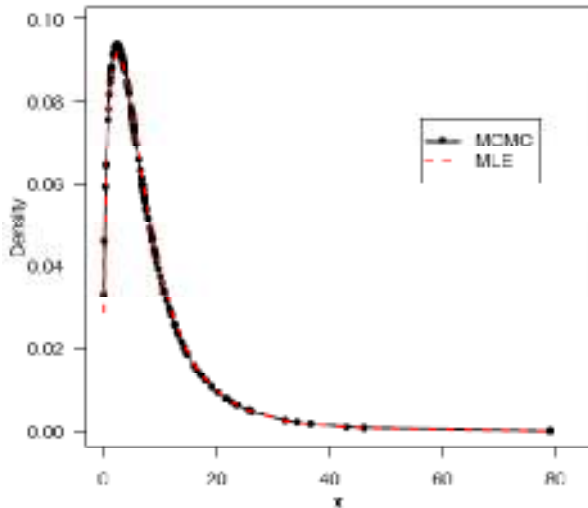


Figure 11: The density functions using MLEs and Bayesian estimates, computed via MCMC

Estimation of reliability and hazard functions

In this section, our main aim is to demonstrate the effectiveness of proposed methodology. For this, we have estimated the reliability function using posterior samples. Since we have an effective MCMC technique, we can estimate any function of the parameters. We have computed the reliability and hazard function at each observed data point for 5000 posterior samples as

$$R(x_i; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}); j = 1, \dots, 5000; i = 1, \dots, 128$$

$$h(x_i; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}); j = 1, \dots, 5000; i = 1, \dots, 128$$

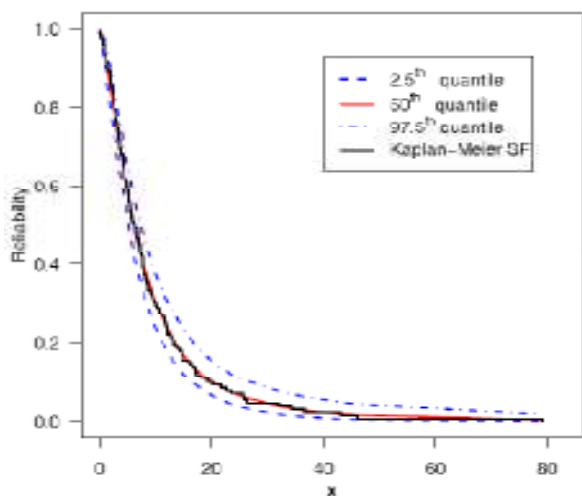


Figure 12: Reliability function estimate using MCMC and Kaplan-Meier estimate

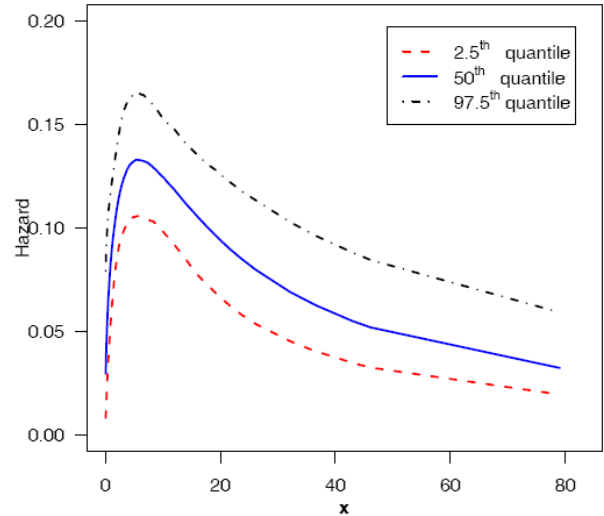


Figure 13: Hazard function estimate using MCMC

The Figure 12, exhibits the estimated reliability function (dashed blue line: 2.5th and 97.5th quantiles; solid red line: 50th quantile) using Bayes estimate (posterior mean) based on MCMC output and the empirical reliability function (black solid line). We have used the Kaplan-Meier estimate of the reliability function to make the comparison more meaningful, Shrestha and Kumar (2013).

The Figure 12 shows that reliability estimate based on MCMC is very close to the empirical reliability estimates. It shows that we have a fairly good model for the given data set. In Figure 13 we have plotted the 2.5th, 50th and 97.5th quantiles of the hazard function.

Estimation of Hazard and Reliability at $X_{(20)}$

The posterior samples may be used to completely summarize the posterior uncertainty about the parameters α , β and λ through a kernel estimate of the posterior distribution.

This is also true of any function of the parameters e.g. reliability and hazard functions. Suppose we wish to give point and interval estimates for reliability and hazard functions at the mission time $t=2.26$ (at the 20th observed data point). We have computed the hazard and reliability functions at mission time $t=2.26$ (at the 20th observed data point) for 5000 posterior samples, using logical function `hrf()` and `reliability()`, (Kumar *et al.* 2011) in OpenBUGS. It can be computed directly using hazard and reliability functions given in (4) and (3), respectively.

$$h(x = 2.26; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}); j = 1, \dots, 5000 \text{ and}$$

$$R(x = 2.26; \alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}); j = 1, \dots, 5000$$

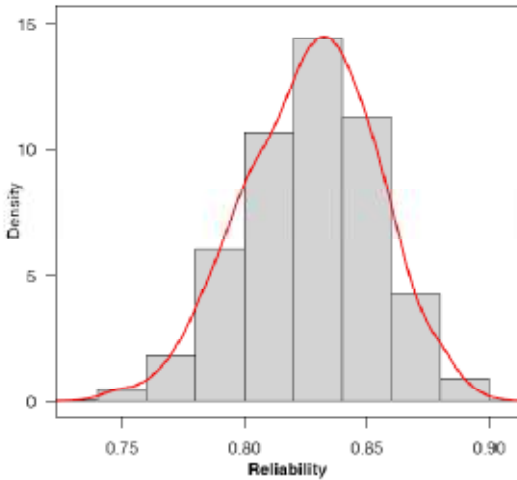


Figure 14(a): Visual summary of reliability function at $t=2.26$

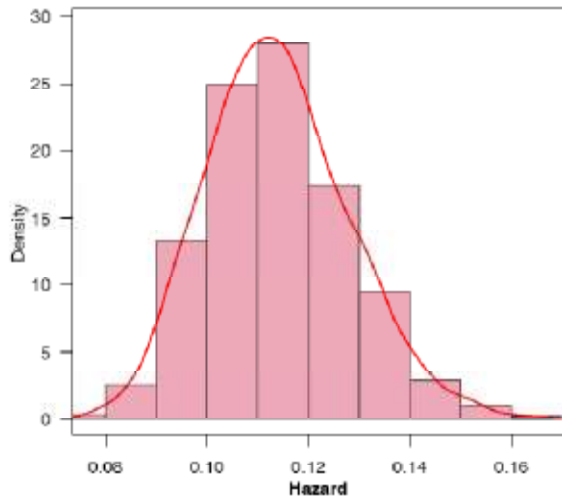


Figure 14(b): Visual summary of hazard function at $t=2.26$

The histogram and marginal posterior density of the reliability and hazard functions are shown in Figure 14(a) and Figure 14(b) respectively based on samples of size 5000. The Gaussian kernel has been used for kernel density estimates.

Table 4: Posterior summary for Reliability and Hazard functions at $t=2.26$

Characteristics	Reliability	Hazard
Mean	0.8269	0.1139
Standard Deviation	0.0272	0.0141
First Quartile (Q_1)	0.8085	0.1040
Median	0.8286	0.1132
Third Quartile (Q_3)	0.8463	0.1227
Mode	0.8327	0.1121
Skewness	-0.2370	0.3486
Kurtosis	-0.1049	0.0954
95% CI	(0.7707, 0.8774)	(0.0893, 0.1434)
95% HPD CI	(0.7754, 0.8796)	(0.0898, 0.1440)

It is evident from Figure 15 the estimates that the marginal distribution of reliability is negatively skewed whereas hazard is positively skewed. The MCMC results of the posterior mean, mode, standard deviation(SD), first quartile, median, third quartile, skewness, kurtosis, 95% symmetric and HPD credible intervals(CI) of reliability and hazard functions are displayed in Table 4.

The ML estimates of reliability and hazard function at $t=2.26$ are computed using invariance property of the MLE. ML estimates $\hat{h}(t=2.26)=0.1139$ and $\hat{R}(t=2.26)=0.8269$. A trace plot is a plot of the iteration number against the value of the draw of the parameter at each iteration. Figure 15 display 5000 chain values for the hazard $h(t=2.26)$ and reliability functions, with their sample median and 95% credible intervals.

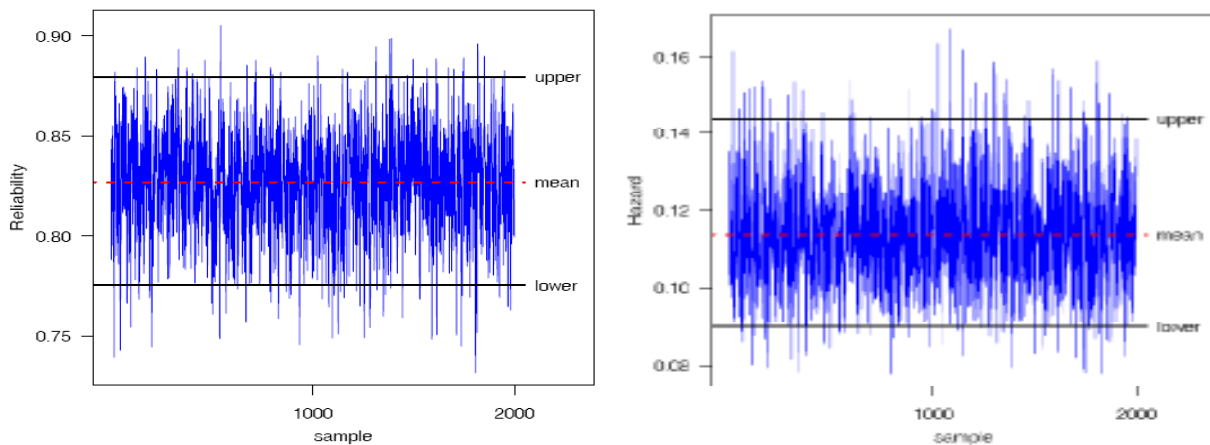


Figure 15: MCMC output of $R(t=2.26)$ and $h(t=2.26)$. Dashed line (...) represents the posterior median and solid lines (-) represent lower and upper bounds of 95% probability intervals (HPD)

7. POSTERIOR PREDICTIVE ANALYSIS

One of the best and most flexible approaches to examining model fit is the use of posterior predictive distributions, (Gelman 2003) and (Gelman *et al.* 2004). The posterior predictive distribution for a model is the distribution of future observations that could arise from the model under consideration. The posterior predictive distribution takes into account both (1) parametric uncertainty and (2) sampling uncertainty from the original model. Parametric uncertainty is captured via the posterior distribution for the parameters, a sample of which is the result of simulation using MCMC methods, (Gupta *et al.* 2008). Sampling uncertainty is captured via the specification of the sampling density for the data. Thus, we can simulate data from the posterior predictive distribution, compare it with the observed data, and, if the simulated data are similar to the observed data, we may conclude the model fits well, (Gelman *et al.* 1996). To evaluate the fit of the posterior distribution of a Bayesian model, we can compare the observed data to the posterior predictive distribution. Gelman *et al.* (1996) propose a diagnostic procedure known as a posterior predictive checking using predictive replicates. Various forms of checking function may be calculated for both new data and actual observations to assess whether the model satisfactorily reproduces certain important aspects of the actual data. Thus such checks go beyond bias and precision.

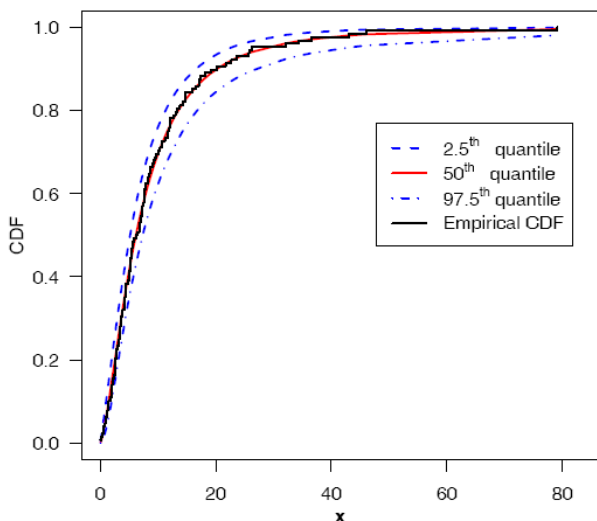


Figure 16: CDF plot

Implementation of posterior predictive simulation is relatively simple, given an MCMC-generated sample of size 2000 from the posterior distribution for the parameters in a model $\underline{\delta} = (\alpha, \beta, \lambda)$, and can often be incorporated as part of the MCMC algorithm itself, Ntzoufras (2009). For each value of $\underline{\delta}$ simulated from the

posterior, we generate a new observation from the sampling distribution for the data, using that parameter value, for every original observation in the sample. Thus, we have

$$x_i^{rep} ; i = 1, \dots, 100 \text{ for each } (\alpha_1^{(j)}, \beta_1^{(j)}, \lambda_1^{(j)}) \quad j = 1, \dots, 2000$$

In fact, we have predicated the entire data set and we have 2000 replications of each. $x_i ; i = 1, \dots, 128$.

The predictive data reflect the expected observations after replicating the experiment in future, having already observed \underline{x} and assuming that the adopted model is true. We view the model-checking as a comparison of the data with the replicated data given by the model, which includes exploratory graphics. In fact statistical graphics provides implicit or explicit model checks. Figure 16 represents the CDF plot. We, therefore, conclude that the GCR distribution is compatible with the given data set. To obtain further clarity on our conclusion for the study of model compatibility, we have considered plotting of density estimates of $(X_{(126)}, X_{(127)} \text{ and } X_{(128)})$ replicated future observations from the model with superimposed corresponding observed data. For this purpose, 2000 samples have been drawn from the posterior using MCMC procedure and then obtained predictive samples from the model under consideration using each simulated posterior sample. The posterior predictive density estimates based on replicated future data sets are shown in Figure 17(a) and Figure 17(b). Figure 17(b) represents the estimates corresponding to $X_{(127)}$ and $X_{(128)}$ predictive observations, whereas the same for $X_{(126)}$ is shown in Figure 17(a).

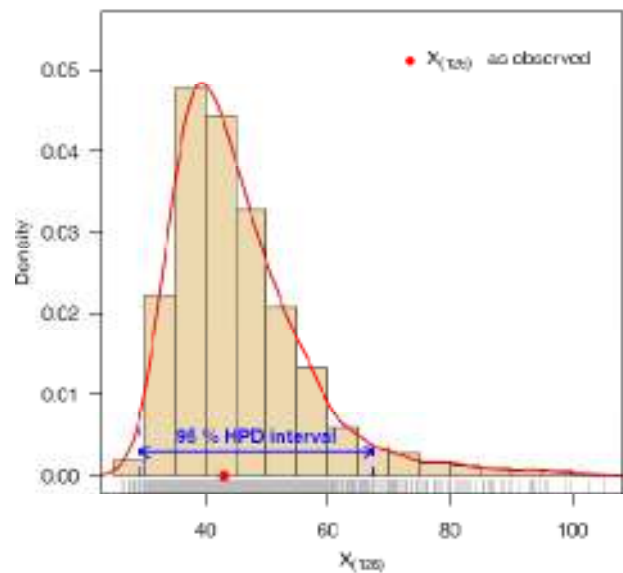


Figure 17(a): Posterior predictive density of $X_{(126)}$

We have also superimposed the 95% HPD credible intervals. The corresponding observed values are marked as (●) on the axis. The HPD intervals include the observed values precisely. Also the Figure 17(a) and Figure 17(b) show, the posterior predictive distributions are centered over the observed values, which indicate good fit. In general, the distribution of replicated data appears to match that of the observed data fairly well.

The MCMC results of the posterior mean, median, mode for $(X_{(1)}, X_{(2)}, X_{(3)}, X_{(80)}, X_{(126)}, X_{(127)}$ and $X_{(128)}$ are displayed in Table 5

	Observed	Mode	Mean	Median
$X_{(1)}$	0.08	0.13	0.15	0.15
$X_{(2)}$	0.20	0.29	0.33	0.32
$X_{(3)}$	0.40	0.41	0.46	0.45
$X_{(80)}$	7.93	8.28	8.34	8.31
$X_{(126)}$	43.01	39.03	45.65	42.97
$X_{(127)}$	46.12	47.16	56.34	52.76
$X_{(128)}$	79.05	68.92	89.07	81.33

Figure 18(left panel) represents the QQ plot of predicted quantiles vs. observed quantiles.

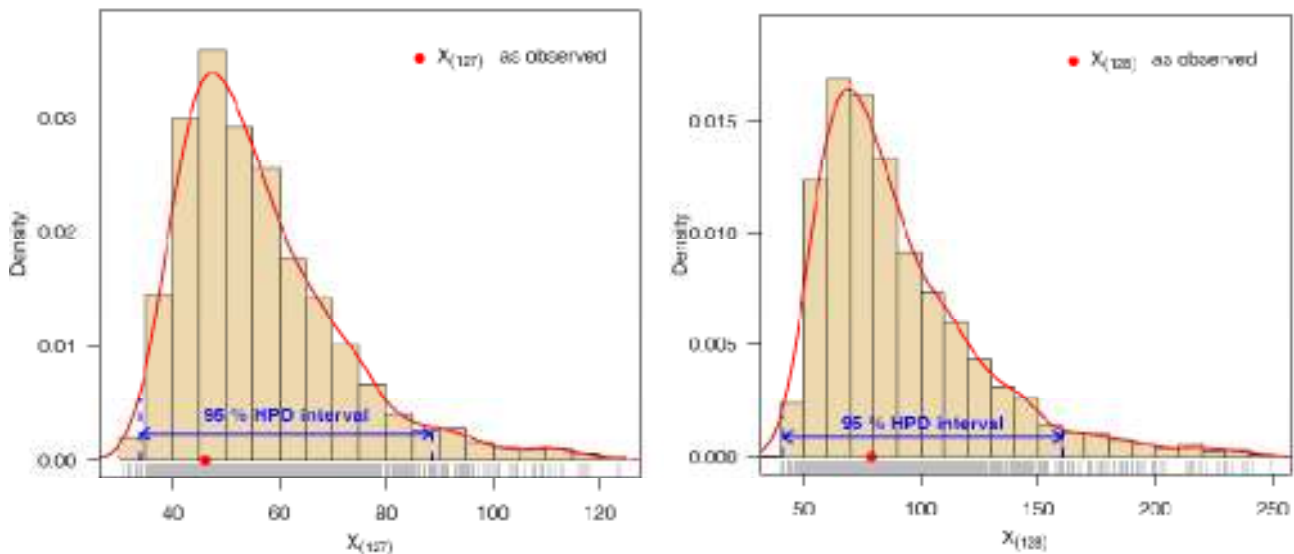


Figure 17(b): Posterior predictive distributions of $X_{(127)}$ and $X_{(128)}$

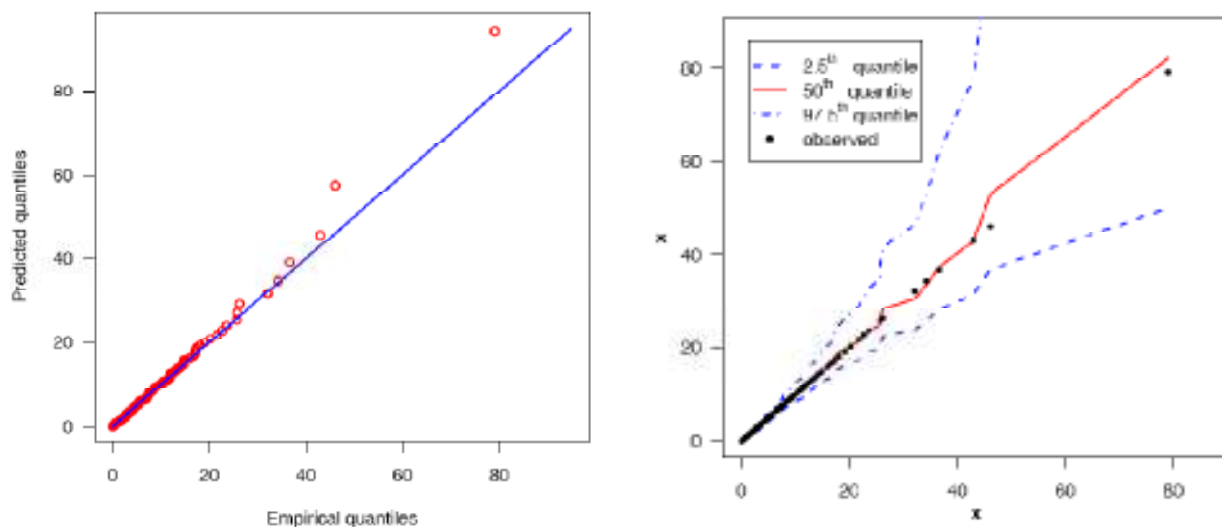


Figure 18: Q-Q plot of predictive quantiles versus empirical quantiles(left panel) Graph of Model fit(right panel)

Figure 18(right panel) exhibits graphical posterior predictive check of the model adequacy, solid line(—) represents the posterior median and dashed lines(...) represent lower and upper bounds of 95% probability intervals, observed data is superimposed.

Model fit assessments based on posterior predictive checks are somewhat too liberal, and posterior predictive checks should not be used for model selection. The results of the posterior predictive simulation indicate that model fits the data particularly well.

8. CONCLUSION

In this paper, the procedure for the maximum likelihood estimation and Bayesian estimation has been discussed. The Bayesian estimation and prediction is considered under independent gamma and uniform priors. The MCMC procedure provides a flexible environment for fitting a wide range of models. The MCMC sample

completely summarizes posterior distribution about the parameters. The MCMC simulation is performed in OpenBUGS. For this we have developed the required computational tool for GCR distribution in OpenBUGS software. The Bayesian analysis is We have used exploratory data analysis techniques for the posterior analysis. We have shown that it is true for any function of the parameters such as hazard function, reliability etc. We have obtained the probability intervals for parameters, hazard and reliability functions. We have presented the model compatibility via the posterior predictive check method. We have applied the developed techniques on a real data set. From the discussion mentioned above, we may conclude that the Bayes procedures discussed in the paper can be recommended for their use for Bayesian estimation and prediction related to generalized compound Rayleigh model.

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