Original Research Article

Relations for single and product moments of generalized order statistics from Sushila distribution

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Abstract In this paper, some recurrence relations for single and product moments of generalized order statistics arising from the Sushila distribution are obtained. Further, various deductions and related results are discussed and identified some of these recurrence relations and other properties of generalized order statistics.

Key Words: Order statistics, k^{th} upper record values, generalized order statistics, Sushila distribution, single moments, product moments, recurrence relations.

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INTRODUCTION

The concept of generalized order statistics (gos) was given by Kamps (1995), which is given as below:

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables (rv) with absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$.

Let $n \in N$, $n \geq 2$, $k \geq 1$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be the parameters such that $\gamma_r = k + n - r + M_r \geq 1$, for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called gos if their joint pdf is given by

$$f_{X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)}(x_1, x_2, \ldots, x_n)$$

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$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i}\ f(x_i)\right)[\bar{F}(x_n)]^{k-1}\ f(x_n) \tag{1.1}$$

on the cone $F^{-1}(0) < x_1 \le \ldots \le x_n < F^{-1}(1)$,

where $\bar{F}(x) = 1 - F(x)$.

Here we may consider two cases:

Case I. $\gamma_i = \gamma_j$ i.e. $m_1 = m_2 = \ldots = m_{n-1} = m$.

Case II. $\gamma_i \ne \gamma_j,\ i,\ j = 1, 2, \ldots, n-1$.

For case I, gos will be denoted as $X(r, n, m, k)$ and its pdf is given by Kamps(1995)

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)),\ \alpha \le x \le \beta \tag{1.2}$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x, y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m\ f(x)\, g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1}\ f(y),\ \alpha \le x < y \le \beta \tag{1.3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i\ ,\ \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \qquad x \in [0,1).$$

For case II, the pdf of $X(r, n, \tilde{m}, k)$ is Kamps and Cramer (2001)

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1}\ f(x)\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1}, \tag{1.4}$$

and the joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,\tilde{m},k)X(s,n,\tilde{m},k)}(x, y) = C_{s-1}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)$$

$$\times\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \tag{1.5}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i\ ,\ \gamma_i = k + n - i + M_i$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r}\frac{1}{(\gamma_j - \gamma_i)},\ 1 \le i \le r \le n$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}, \; r+1 \le i \le s \le n.$$

It may be noted that $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$

$$a_i(r) = \frac{(-1)^{r-i}}{(r-1)!(m+1)^{r-1}} \binom{r-1}{r-i}$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(s-r-1)!(m+1)^{s-r-1}} \binom{s-r-1}{s-i}.$$

Consequently the pdf of $X(r,n,\tilde{m},k)$ and the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ reduce to the pdf of $X(r,n,m,k)$ and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, respectively. Several models of ordered random variables such as order statistics and record value can be seen as special cases of gos. If $m=0$ and $k=1$, then $X(r,n,m,k)$ reduces to the r^{th} order statistic $X_{r:n}$ David and Nagaraja (2003). If $m=-1$ and $k=1$, then $X(r,n,m,k)$ is the r^{th} record values from an infinite sequence of $iid\,rv's$ Ahsanullah (1995). Other special cases are k^{th} record values $(m=-1,\, k\in \mathbb{N})$, Dziubdziela and Kopociński (1976), sequential order statistics $((\gamma_i = n-i+1)\beta_i; \beta_1, \beta_2, \ldots, \beta_n > 0)$ and order statistics with non–integral sample size ($m=0,\; k=\alpha-n+1,\; \alpha=n,$ Stigler (1977), Rohatgi and Saleh (1988)).

Many authors utilized the concept of *gos* in their studies. References may be made to Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001),Ahmad and Fawzy (2003), Ahmad (2007), Khan *et al.* (2007),Athar *et al.* (2012),Saran *et al.*(2015), Khan and Khan (2016) among others. For textbook reference, one may referrred to Ahsanullah (1995), Ahsanullah and Nevzorov (2001), Kamps (1995) and Arnold *et al.* (1992). In this paper, we have obtained the recurrence relation of gos arising from the Sushila distribution.

A random variable(rv) X is said to have a Sushila distributionShanker *et al.*(2013) if its df asgiven by

$$F(x) = 1 - \frac{\lambda(\sigma+1)+\sigma x}{\lambda(\sigma+1)} e^{-\frac{\sigma}{\lambda}x}; \;\; x>0,\, \sigma>0,\, \lambda>0 \tag{1.6}$$

and thecorresponding pdf is given by

$$f(x) = \frac{\sigma^2}{\lambda(1+\sigma)}\left(1+\frac{x}{\lambda}\right) e^{-\frac{\sigma}{\lambda}x}; \; x>0,\; \sigma>0,\; \lambda>0. \tag{1.7}$$

The Sushila distribution given in (1.7) was introduced by Shanker *et al.* (2013). At $\lambda=1$, it reduces to Lindley distribution (Lindley,1958)) having pdf as given by

$$f(x) = \frac{\sigma^2}{1+\sigma}(1+x)e^{-\sigma x}; \; x>0, \sigma>0, \tag{1.8}$$

(Ghitany *et al.*, 2008) have explored some interesting properties of this distribution and showed that Lindley distribution gives better lifetime model than the exponential distribution in applications. Sankaran (1970) introduced the discrete Poisson-Lindley distribution after mixing Poisson and Lindley distribution. Zakarzadeh and Dolati (2009) introduced the generalization of Lindley distribution having three parameters.

It is observed that Lindley distribution is a particular case of (1.7). The pdf (1.7) can be shown as a mixture of exponential $\left(\frac{\sigma}{\lambda}\right)$ and gamma $\left(2,\frac{\sigma}{\lambda}\right)$ distribution as follows:

$$f(x;\sigma,\lambda) = pf_1(x) + (1-p)\, f_2(x) \tag{1.9}$$

The relation between (1.6) and (1.7), we have

$$\left(\lambda\left(1+\sigma\right)+\sigma x\right) fx=\sigma^2\left(1+\frac{x}{\lambda}\right)\left(1-F\left(x\right)\right) \tag{1.10}$$

The relation (1.10) is used for obtaining the recurrence relations for moments of *gos* from Sushila distribution. In this paper, we have established recurrence relations for single and product moments of generalized order statistics from Sushila distribution. This paper comprise three sections. In Section 2, we have established the recurrence relation based on single moment of generalized order statistics from Sushila distribution. In Section 3, we have obtained the recurrence relation based on product moment of generalized order statistics from Sushila distribution.

RECURRENCE RELATIONSFOR SINGLE MOMENTS

Theorem 2.1: Let X be a non-negative continuous random variable and follows Sushila distribution given in (1.7). For Case II $\gamma_i \neq \gamma_j, i\neq j \in (1,2,\ldots,n-1)$, $k=1,2,\ldots$, $n\in$, $1\le r\le n$, $l=0,1,2,\ldots$

$$\begin{aligned}\lambda\left(1+\sigma\right)E\left[X^{l}\left(r,n,\tilde{m},k\right)\right]+\sigma E\left[X^{l+1}\left(r,n,\tilde{m},k\right)\right] &= \frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{l+1}\left(r-1,n,\tilde{m},k\right)\right]\right]\\ &+\frac{\sigma^2\gamma_r}{\lambda\left(l+2\right)}\left[E\left[X^{l+2}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{l+2}\left(r-2,n,\tilde{m},k\right)\right]\right]\end{aligned} \tag{2.1}$$

Proof. We have

$$\lambda\left(1+\sigma\right)E\left[X^{l}\left(r,n,\tilde{m},k\right)\right]+\sigma E\left[X^{l+1}\left(r,n,\tilde{m},k\right)\right]$$

$$=c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^{r}a_i\left(r\right)(1-F(x))^{\gamma_i-1}\right)\left(\lambda\left(1+\sigma\right)+\sigma x\right) fx\,dx$$

On using equation (1.10), we have

$$=c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^{r}a_i\left(r\right)(1-F(x))^{\gamma_i-1}\right)\sigma^2\left(1+\frac{x}{\lambda}\right)\left(1-F\left(x\right)\right)dx$$

$$\begin{aligned}&=\sigma^2 c_{r-1}\int_0^\infty x^l\sum_{i=1}^{r}a_i\left(r\right)(1-F(x))^{\gamma_i-1}\left(1-F\left(x\right)\right)dx\\ &+\frac{\sigma^2}{\lambda}c_{r-1}\int_0^\infty x^{l+1}\sum_{i=1}^{r}a_i\left(r\right)(1-F(x))^{\gamma_i-1}\left(1-F\left(x\right)\right)dx\end{aligned} \tag{2.2}$$

$$\begin{aligned}&=\sigma^2 c_{r-1}\int_0^\infty x^l\sum\nolimits_{i=1}^{r}a_i\left(r\right)\left(1-F(x)\right)^{\gamma_i}dx\\ &+\frac{\sigma^2}{\lambda}c_{r-1}\int_0^\infty x^{l+1}\sum\nolimits_{i=1}^{r}a_i\left(r\right)\left(1-F(x)\right)^{\gamma_i}dx\end{aligned}$$

$$\lambda\left(1+\sigma\right)E\left[X^{l}\left(r,n,\tilde{m},k\right)\right]+\sigma E\left[X^{l+1}\left(r,n,\tilde{m},k\right)\right]=I+II$$

Now,

$$I=\sigma^2 c_{r-1}\int_0^\infty x^l\left(\sum_{i=1}^{r}a_i\left(r\right)(1-F(x))^{\gamma_i}\right)dx$$

Integrating by parts treating x^l as integration and rest of the part for differentiation.

$$=\sigma^2 c_{r-1}\left[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\cdot\frac{x^{l+1}}{l+1}\Bigg|_0^\infty-\int_0^\infty x^l\left(\frac{d}{dx}\left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right)\right)\left(\int x^l dx\right)dx\right]$$

$$=\frac{\sigma^2}{l+1}c_{r-1}\int_0^\infty x^{l+1}\left(\frac{d}{dx}\left(\sum_{i=1}^{r} a_i(r)\ (1-F(x))^{\gamma_i}\right)\right)dx=\frac{\sigma^2}{l+1}c_{r-1}\int_0^\infty x^{l+1}\sum_{i=1}^{r} a_i(r)\gamma_i(1-F(x))^{\gamma_i-1}\ f(x)dx$$

on using (2.1) and $c_{r-1}=\gamma_r c_{r-2}$, we have

$$=\frac{\sigma^2}{l+1}c_{r-1}\int_0^\infty x^{x+1}\left[\gamma_r\left\{c_{r-1}\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}-c_{r-2}\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i}\right\}\right]fx\,dx$$

$$-\frac{\sigma^2\gamma_r}{l+1}\left[c_{r-1}\int_0^\infty x^{x+1}\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\ fx\,dx-c_{r-2}\int_0^\infty\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i}\ fx\,dx\right]$$

Therefore

$$I=\frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}(r,n,\tilde{m},k)\right]-E\left[X^{l+1}(r-1,n,\tilde{m},k)\right]\right]$$

similarly second integral is given by

$$II=\frac{\sigma^2}{\lambda}c_{r-1}\int_0^\infty x^{l+1}\sum_{i=1}^{r} a_i(r)(i-F(x))^{\gamma_i}\ dx\qquad=\frac{\sigma^2\gamma_r}{\lambda(l+2)}\left[E\left[X^{l+2}(r,n,\tilde{m},k)\right]-E\left[X^{l+2}(r-1,n,\tilde{m},k)\right]\right]$$

substituting the value of the integral I and II in equation no(2.3), we have

$$\lambda(1+\sigma)E\left[X^l(r,n,\tilde{m},k)\right]+\sigma E\left[X^{l+1}(r,n,\tilde{m},k)\right]$$

$$=\frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}(r,n,\tilde{m},k)\right]-E\left[X^{l+1}(r-1,n,\tilde{m},k)\right]\right]$$

$$+\frac{\sigma^2\gamma_r}{\lambda(l+2)}\left[E\left[X^{l+2}(r,n,\tilde{m},k)\right]-E\left[X^{l+2}(r-1,n,\tilde{m},k)\right]\right]$$

and hence the result.

Corollary 2.1: For $m_1=m_2=\ldots=m_{n-1}=m$, the recurrence relation for single moments of *gos* from Sushila distribution has the form

$$\lambda(1+\sigma)E\left[X^l(r,n,m,k)\right]+\sigma E\left[X^{l+1}(r,n,m,k)\right]$$

$$=\frac{\sigma^2\gamma_r}{(l+1)}\left[E\left[X^{l+1}(r,n,m,k)\right]-E\left[X^{l+1}(r-1,n,m,k)\right]\right]$$

$$+\frac{\sigma^2\gamma_r}{\lambda(l+2)}\left[E\left[X^{l+2}(r,n,m,k)\right]-E\left[X^{l+2}(r-2,n,m,k)\right]\right]\qquad(2.3)$$

Remark 2.1: At $\lambda=1$ in (2.3) we get the recurrence relation for single moments of generalized order statistics from Lindley distribution.

Remark 2.2: Recurrence relation for single moments of order statistics (at $m=0,\ k=1$) from Sushila distribution is

$$\lambda(1+\sigma)E\left[X_{r:n}^{j}\right]+\sigma E\left[X_{r:n}^{j+1}\right]$$

$$=\frac{\sigma^2(n-r+1)}{(l+1)}\left[E\left[X_{r:n}^{l+1}\right]-E\left[X_{r-1:n}^{l+1}\right]\right]$$

$$+\frac{\sigma^2(n-r+1)}{\lambda(l+2)}\left[E\left[X_{r:n}^{l+2}\right]-E\left[X_{r-2:n}^{l+2}\right]\right] \quad (2.4)$$

Remark.2.3: Recurrence relation for single moments of k^{th} upper record $(m=-1)$ from Sushila distribution is

$$\lambda(1+\sigma)E\left[X_{U(r)}^{(k)}\right]^{j}+\sigma E\left[X_{U(r)}^{(k)}\right]^{j+1}$$
$$=\frac{\sigma^2 k}{(l+1)}\left[E\left[X_{U(r)}^{(k)}\right]^{l+1}-E\left[X_{U(r-1)}^{(k)}\right]\right]^{l+1}$$
$$+\frac{\sigma^2 k}{\lambda(l+2)}\left[E\left[X_{U(r)}^{(k)}\right]^{l+2}-E\left[X_{U(r-2)}^{(k)}\right]^{l+2}\right]$$

RECURRENCE RELATIONS FOR PRODUCT MOMENTS

In this section, the recurrence relation for product moments of *gos* from Sushila distribution has been obtained. Particular cases for recurrence relations of order statistics and k^{th} upper record are also discussed.

Theorem 3.1: Let X be a non-negative continuous random variable and follows Sushila distribution given in (1.7). Let case II be satisfied i.e. $\gamma_i \neq \gamma_j, i \neq j \in (1,2,\ldots,n-1)$. For Sushila distribution as given in (1.7) and $k \geq 1, n \in N, \tilde{m} \in R, 1 \leq r < s \leq n, s-r \geq 2,$ and $u, v = 0,1,2,\ldots$

$$\lambda(1+\sigma)E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]+\sigma E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$
$$=\frac{\sigma^2\gamma_s}{(v+1)}\left[E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]-E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s-1,n,\tilde{m},k)\right]\right]$$
$$+\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[E\left[X^u(r,n,\tilde{m},k)X^{v+2}(s,n,\tilde{m},k)\right]-E\left[X^u(r,n,\tilde{m},k)X^{v+2}(s-1,n,\tilde{m},k)\right]\right]$$

(3.1)

Proof: We have

$$\lambda(1+\sigma)E\left[X^u(r,n,\tilde{m},k)X^{v}(s,n,\tilde{m},k)\right]+\sigma E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$
$$=\int_0^\infty x^u\left\{\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right\}\frac{f(x)}{(1-F(x))}I(x)\,dx \quad (3.2)$$

where $I(x)$ is given by

$$I(x)=c_{s-1}\int_x^\infty y^v\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\frac{fy}{(1-F(y))}dy$$

On substituting f(y) in above equation

$$=\sigma^2 c_{s-1}\int_x^\infty y^v\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\left(1+\frac{y}{\lambda}\right)dy$$

$$I(x)=I_1(x)+I_2(x)$$

Where $I_1(x)$ is given by

$$I_1(x)=\frac{\sigma^2\gamma_s}{v+1}\left[\left\{c_{s-1}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(y)}\right)^{\gamma_i}\right.\right.$$
$$\left.\left.-c_{s-2}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(y)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}dy \quad (3.5)$$

Similarly, $I_2(x)$

$$I_2(x)=\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[\left\{c_{s-1}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right.\right.$$
$$\left.\left.-c_{s-2}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}dy \quad (3.6)$$

Therefore

$I(x)=I_1(x)+I_2(x)$

On substituting $I(x)$ in (3.2), we get

$$\lambda(1+\sigma)E\left[X^u(r,n,\tilde{m},k)X^v(s,n,\tilde{m},k)\right]+\sigma E\left[X^u(r,n,\tilde{m},k)X^{v+1}(s,n,\tilde{m},k)\right]$$
$$=\int_0^\infty x^u\left[\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right]\frac{f(x)}{(1-F(y))}I(x)\,dx$$
$$=\int_0^\infty x^u\left\{\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right\}\frac{f(x)}{(1-F(y))}\left[\frac{\sigma^2\gamma_s}{v+1}\left[\left\{c_{s-1}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right.\right.\right.$$
$$\left.\left.-c_{s-2}\int_x^\infty y^{v+1}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}dy$$
$$+\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[\left\{c_{s-1}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\right.\right.$$
$$\left.\left.\left.-c_{s-2}\int_x^\infty y^{v+2}\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\right]\frac{f(y)}{(1-F(y))}\right]dy$$

and hence the result given in (3.1).

Corollary 3.1: For $m_1=m_2=\ldots=m_{n-1}=m$, the recurrence relation for product moments of *gos* from Sushila distribution has the form

$$\lambda(1+\sigma)E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right]+\sigma E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right]$$
$$=\frac{\sigma^2\gamma_s}{(v+1)}\left[E\left[X^u(r,n,m,k)X^{v+1}(s,n,m,k)\right]-E\left[X^u(r,n,m,k)X^{v+1}(s-1,n,m,k)\right]\right]$$
$$+\frac{\sigma^2\gamma_s}{\lambda(v+2)}\left[E\left[X^u(r,n,m,k)X^{v+2}(s,n,m,k)\right]-E\left[X^u(r,n,m,k)X^{v+2}(s-1,n,m,k)\right]\right] \quad (3.6)$$

Remark 3.1: At $\lambda = 1$ in (3.4) we get the recurrence relation for product moments of generalized order statistics from Lindley distribution.

Remark 3.2: Recurrence relation for product moments of order statistics (at $m = 0,\ k = 1$) from Sushila distribution is

$$\lambda(1+\sigma)E\left[X_{r:n}^{u}X_{r:n}^{v+1}\right]+\sigma E\left[X_{r:n}^{u}X_{r:n}^{v+1}\right]$$

$$=\frac{\sigma^{2}(n-s+1)}{(v+1)}\left[E\left[X_{r:n}^{u}X_{s:n}^{v+1}\right]-E\left[X_{r:n}^{u}X_{s-1:n}^{v+1}\right]\right]$$

$$+\frac{\sigma^{2}(n-s+1)}{\lambda(v+2)}\left[E\left[X_{r:n}^{u}X_{s:n}^{v+2}\right]-E\left[X_{r:n}^{u}X_{s-1:n}^{v+2}X^{u}(r,n,\tilde{m},k)\right]\right] \tag{3.7}$$

Remark 3.2: Recurrence relation for product moments of k^{th} upper record $(m=-1)$ from Sushila distribution.

REFERENCE

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