Original Research Article

A note on risk efficiency and regret of sequential procedure of mean

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Abstract

The Problem of minimum risk point estimation of the mean of the two paramater exponentiated inverted Weibull distribution is taken up. Consideration is given to the squared-error loss function and linear cost of sampling. The failure of fixed sample size procedure is established and a sequential procedure is developed. Some asymptotic properties are obtained for the sequential procedure.
Key Words: Sequential procedure, loss function, Risk efficiency and Regret.

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INTRODUCTION

Flaih *et al* (2012) proposed the two paramater exponentiated inverted Weibull distribution (EIWD). This distribution is a generalization to the inverted Weibull distribution through adding a new shape parameter θ € R+ by exponentiation to distribution function F. Let us consider a sequence $\{X_i\}; i = 1,2,3, \ldots$ of independently distributed random variable then from two parameter exponentiated inverted Weibull distribution with the p.d.f given as

$f(x;\theta,\beta) = \theta\beta x^{-(\beta+1)} exp(-\theta x^{-\beta}); x > 0, \theta > 0, \beta > 0$. (1.1)

For θ=1, it represents the standard inverted Weibull distribution and for β=1, it represents the exponentiated inverted exponential distribution.

From equation(1.1), the mean of the distribution is given by $\emptyset = E(X_i) = \theta^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)$ and $Var(X_i) = \theta^{\frac{2}{\beta}}\Gamma\left(1-\frac{2}{\beta}\right) - \theta^{\frac{2}{\beta}}\left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2$.

This distribution is really used in literature and has its applications in estimating the parameters so this sequential Procedure is considered instead of fixed sample size in this paper. The pioneer work of sequential test is proposed by Wald (1947), he developed sequential probability ratio test(SPRT) for testing hypothesis against a simple alternative. Further, in recent decades many authors has developed sequential procedure for point and interval estimation. For some citation, one may refers to Anscombe (1952), Woodroofe (1977), Zacks (1971), Nagao (1980), Chaturvedi (1987), Chaturvedi, Pandey and Gupta (1991), Chaturvedi and Shukla (1990), Mukhopadhyay and Pepe (2006), Uno, Isogai and Lim (2004), Roughani and Mohmoudi (2015) and others. The purpose of this note is two-fold. The problem of minimum risk point estimation of the mean(∅) of exponentiated inverted Weibull distribution under the square error loss function and linear cost of sampling is considered. It has been proved that the failure of the fixed sample size procedures to handle the estimation problem. purely sequential procedure are developed to tackle the situations. In section 2, the set-up of the estimation problem has been described and proved the failure of fixed sample size procedure to deal with them. In section 3, the sequential procedure for the point estimation of the mean of two parmater exponentiated inverted Weibull

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distribution has been proposed and the asymptotic Risk efficiency and regret for the given sequential procedure is achieved in the paper.

SET-UP OF THE ESTIMATION PROBLEM

Having record a sample $X_1, X_2, \ldots, X_n$ of size n, let us define $\bar{X}_n = \frac{\sum_{i=1}^{n} X_i}{n}$, as the estimator for $\emptyset$. Let the Loss-incurred in estimating $\emptyset$ by $\bar{X}_n$ is of the form

$L_n(C) = A(\bar{X}_n - \emptyset)^2 + nC$, (2.1)

where A(>0) is known weight, C(>0) is the cost function. Form the loss(2.1), the risk comes out to be

$$R_n(C) = \frac{A}{n}\theta^{\frac{2}{\beta}}\left[\Gamma\left(1-\frac{2}{\beta}\right) - \left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2\right] + nC. \quad (2.2)$$

Treating n as a continuous variable, the value n^* of n, which minimize the risk is

$$n^* = \left(\frac{A}{c}\right)^{\frac{1}{2}} \emptyset \left[\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2} - 1\right]^{\frac{1}{2}}. \quad (2.3)$$

and the corresponding risk when n=n^*, weget,

$R_{n^*}(C) = 2n^*C$. (2.4)

Once again, since n^* depends on θ, in the absence of any knowledge about θ, no fixed sample size procedure minimizes the risk simultaneously for all values of θ. So, to handle this problem in the following section we adopt sequential procedure.

SEQUENTIAL PROCEDURE FOR THE POINT ESTIMATION OF MEAN

Without loss of generality, we assume β is known i.e. β=4, we rewrite (n^*) and $R_{n^*}(C)$ as,

$$n^* = K\left(\frac{A}{c}\right)^{\frac{1}{2}} \emptyset_0, \quad (2.5)$$

where, $\emptyset_0$ is the value of mean at β=4 and $K = \left[\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2} - 1\right]^{\frac{1}{2}}$.

$R_{n^*}(C) = 2K\sqrt{AC}\emptyset_0$. (2.6)

The stopping time $N \equiv N(C)$ is the first positive integer $n \geq m(\geq 2)$ defined by

$$N = inf\left[n \geq m; n \geq \left(\frac{A}{c}\right)^{\frac{1}{2}}\left[\theta^{\frac{2}{\beta}}\Gamma\left(1-\frac{2}{\beta}\right) - \left[\Gamma\left(1-\frac{1}{\beta}\right)\right]^2\right]^{\frac{1}{2}}\right] \quad (2.7)$$

$$N = inf\left[n \geq m; n \geq \left(\frac{A}{c}\right)^{\frac{1}{2}} \emptyset_0 K\right] \quad (2.8)$$

where, m being the starting sample size. Use $\bar{X}_N$ to estimate $\emptyset_0$.

Following Starr (1966a,1996b), we define the Risk-efficiency of the sequential procedure is given by

$$R_e(C) = \frac{R_N(C)}{R_{n^*}(C)} \quad (2.9)$$

Also we define Regret as,

$R_g(C) = R_N(C) - R_{n^*}(C)$, (2.10)

where, $R_N(C)$ is Risk associated with the sequential procedure, i.e.

$$R_N(C) = E\left(L(\emptyset_0, \bar{X}_N)\right)$$
$$= A[E((\bar{X}_N - \emptyset_0)^2)] + C\,E(N). \quad (2.11)$$

The following theorem based on Risk efficiency and Regret as follows is proved.

Theorem 1. For the stopping rule define in (2.8) and all $\emptyset_0$,

$$\lim_{C\to 0} R_e(C) = 1. \quad (2.12)$$

Proof. Rewrite the (2.8) and using $S_n = \sum_{i=1}^{n} X_i$,

$$N = inf\left[n \geq 1; n \geq \left(\frac{A}{c}\right)^{\frac{1}{2}} \emptyset_0 K\right]$$

$$N = inf\left[n \geq 1; n \geq \left(\frac{A}{c}\right)^{\frac{1}{2}} \bar{X}_n K\right]$$

$$N = inf\left[n \geq 1; n^2 \geq K\left(\frac{A}{c}\right)^{\frac{1}{2}} \sum_{i=1}^{n} X_i\right]$$

$$N = inf\left[n \geq 1; n^2 \geq K\left(\frac{A}{c}\right)^{\frac{1}{2}} S_n\right]$$

$$N = inf\left[n \geq 1; S_n \leq \frac{n^2}{K\left(\frac{A}{c}\right)^{\frac{1}{2}}}\right]. \quad (2.13)$$

From Wald's Lemma for cumulative sum,

$E[(S_N - N\emptyset_0)^2] = K^2\emptyset_0^2 E(CN)$.

From (2.11), weget,

$$R_N(C) = A[E((\bar{X}_N - \emptyset_0)^2)] - C\,E(N) + 2C\,E(N)$$
$$= A\left[E\left(\left(\frac{S_N}{N} - \emptyset_0\right)^2\right)\right] - C\,E(N) + 2C\,E(N); \; \bar{X}_N = \frac{S_N}{N}$$
$$= E\left((S_N - N\emptyset_0)^2 \frac{A}{N^2}\right) - C\,E(N) + 2C\,E(N)$$
$$= E\left((S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right) + 2CE(N). \quad (2.14)$$

Substitute (2.6), (2.11) in (2.9),weget,

$$R_e(C) = \frac{E\left((S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right) + 2CE(N)}{2K\sqrt{AC}\emptyset_0}$$

$$= E\left((S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right)\left(2K\sqrt{AC}\emptyset_0\right)^{-1} + E\left(\frac{N}{\left(\frac{A}{C}\right)^{\frac{1}{2}}\emptyset_0 K}\right). \quad (2.15)$$

It can be seen that $\frac{N}{\left(\frac{A}{C}\right)^{\frac{1}{2}}} \longrightarrow \emptyset_0 K$ almost surely(a.s) as $C \longrightarrow 0$.

The result follows from Gut(1974) that

$\left[\left(\left(\frac{A}{C}\right)^{-\frac{1}{2}}N\right)^4 : A \geq 1\right]$, is uniformly integrable. (2.16)

Using (2.16) and Domianted Convergence theorem,

$\lim_{C\to 0} E\left(\frac{N}{\left(\frac{A}{C}\right)^{\frac{1}{2}}\emptyset_0 K}\right)$=1. (2.17)

From (2.15) and (2.17), we conclude that the result follows if we can prove that

$E\left(\left(\frac{A}{C}\right)^{-\frac{1}{2}}(S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right)$=o(1) as $C \longrightarrow 0$. (2.18)

Using Holder's Inequality,

$E\left|\left(\frac{A}{C}\right)^{-\frac{1}{2}}(S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right| \leq$

$E^{\frac{1}{2}}\left|\left(\frac{A}{C}\right)^{-\frac{1}{4}}(S_N - N\emptyset_0)\right|^4 E^{\frac{1}{2}}\left|\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right|^2$. (2.19)

From (2.16) and Lemma 5 of Chow and Yu (1981),

$\left[\left|\left(\frac{A}{C}\right)^{-\frac{1}{4}}(S_N - N\emptyset_0)\right|^4 : A \geq 1\right]$ is uniformly integrable. (2.20)

From the definition of N at (9), $N^{-2}\left(\frac{A}{C}\right) \leq (S_N K)^{-1}\left(\frac{A}{C}\right)^{\frac{1}{2}}$.

Hence, using the dominated ergodic theorem of Marcinkiewicz and Zygmund[see Chow and Teicher(1978, p.35)],

$\left[\left(\left(\frac{A}{C}\right)^{\frac{1}{2}}N^{-4}\right) : A \geq 1\right]$ is uniformly integrable. (2.21)

Since $\left(\frac{A}{C}\right)N^{-2} \longrightarrow \emptyset_0 K^{-1}$(a.s.) as as $C \longrightarrow 0$, using (2.20) and (2,21), we obtained from (2.19) that

$\left(\frac{A}{C}\right)^{\frac{1}{2}} E\left|(S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right| \leq$ o(1) $E^{\frac{1}{2}}\left|\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right|^2$

= o(1) as $C \longrightarrow 0$.

In the next theorem, we prove the bounded nature of the 'Regret'.

Theorem 2: For the Sequential Procedure (2.8),

$\lim_{C\to 0} R_g(C) = o\left(\left(\frac{A}{C}\right)^{\frac{1}{2}}\right)$.

Proof: $R_g(C) = R_N(C) - R_{n^*}(C)$,

From (2.14) and (2.6), we obtain as,

$R_g(C) = E\left((S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right)$+2CE(N)-$2K\sqrt{AC}\emptyset_0$

=

$E\left[(S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right] + 2\left(\frac{A}{C}\right)^{\frac{1}{2}} E\left[N\left(\frac{A}{C}\right)^{-\frac{1}{2}} - K\emptyset_0\right]$. (2.22)

On using Holder's inequality, weget

$E\left|(S_N - N\emptyset_0)^2\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right| \leq \left(\frac{A}{C}\right)^{\frac{1}{2}} E^{\frac{1}{2}}\left|\left(\frac{A}{C}\right)^{-\frac{1}{4}}(S_N - N\emptyset_0)\right|^4 E^{\frac{1}{2}}\left|\left(\frac{A}{N^2} - K^{-2}\emptyset_0^{-2}\right)\right|^2$. (2.23)

Utilizing (2.20), (2.21)and (24), we obtained form (2.22) as $C \longrightarrow 0$.

REFERENCES

1. Anscombe, F. J. (1952): Large sample theory of sequential estimation. Proc. Cambridge Philos. Soc., 48, 600-607.
2. Chaturvedi, A. (1987): Sequential point estimation of the mean of a log-normal distribution. Jour. Statist. Res., 21, 21-27.
3. Chaturvedi, A., Pandey, S.K. and Gupta,M.(1991). On a class of asymptotically risk-efficiency sequential procedures. Scandinavian Actur.Jour. 1, 87-96.
4. Chaturvedi, A. and Shukla, P. S. (1990): Sequential point estimation of location parameter of a negative exponential distribution. Jour. Indian Statist. Assoc., 28, 41-50.
5. Chow, Y. S. and Teicher, H. (1978): Probability Theory. Spinger verlag, New York.
6. Chow, Y. S. and Yu, K. F. (1981): The performance of a sequential procedure for estimation of the mean. Ann. Statist., 9, 184-189.
7. Flaih, A., Elsalloukh, H., Mendi, E. and Milanova, M., (2012). The Exponentiated Inverted Weibull Distribution. Applied Mathematics andInformation Sciences, Vol. 6, 167-171.
8. Gut, A. (1974): On the moments and limit distribution of some first passage time. Ann. Prob., 2, 308-277.
9. Mahmoudi, E. and Roughani, Gh. (2015). Bounded Risk Estimation of the Scale Parameter of a Gamma Distribution in a Two-Stage Sampling Procedure. Sequential Analysis 34: 25-38.
10. Mukhopadhyay, N. and Pepe, W. (2006). Exact Bounded Risk Estimation When the Terminal Sample Size and Estimator Are Dependent: The Exponential Case. Sequential Analysis 25: 85–101.
11. Mukhopadhyay, N. and Bapat, S. R. (2017). Purely Sequential Bounded-Risk Point Estimation of the Negative Binomial Mean under Various Loss Functions: One-Sample Problem, Annals of Institute of Statistical Mathematics.
12. Nagao, H. (1980): On stopping times of sequential estimations of the mean of a Log-normal distribution. Ann. Ins. Statist. Math., 32, 369-375.
13. Starr, N. (1966 a): The performance of a sequential procedure for the fixed-width interval estimation of the mean. Ann. Math. Statist., 37, 36-50.

14. Starr, N. (1966 b): On the asymptotic efficiency of a sequential procedure for estimating the mean. Ann. Math. Statist., 37, 1137-1185.
15. Uno, C., Isogai, E., and Lim, D. L. (2004). Sequential Point Estimation of a Function of the Exponential Scale Parameter. Austrian Journal of Statistics 33: 281-291.
16. Wald, A. (1947). Sequential Analysis, New York:Wiley.
17. Woodroofe, M. (1977): Second-order approximations for sequential point and interval estimation. Ann. Statist., 5, 984-995.
18. Zacks, S. (1971): The Theory of Statistical Inferential. John Wiley and Sons, New York.

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