A Few Inherent Attributes of One Dimensional Nonlinear Map

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Abstract: Here, we consider a one dimensional non-linear cubic map to find out a few inherent attributes i.e., fixed points, periodic points, bifurcation values of period 2ⁿ, n = 0, 1, 2, 3, 4, ................. We use suitable numerical methods and have shown how the period doubling bifurcation points ultimately converge to the Feigenbaum constant. We have calculated Feigenbaum δ value also. We have further verified our findings with the help of bifurcation diagram, Lyapunov exponent, time series analysis of the map. Computer software package ‘Mathematica’ and ‘C-program’ are used prudentially to implement numerical algorithms for our purpose.

Keywords: Fixed points, Periodic points, Bifurcation points, Feigenbaum constant, Lyapunov exponent, Time series.

1. Introduction

The nonlinear dynamics of physical systems are analyzed by obtaining discrete models. Mathematically, models are represented by maps. Maps arise in various ways. The ways are--------

i) As tools from analyzing differential equations.
   ii) As models of natural phenomenon.
   iii) As simple examples of chaos.

Maps are capable of more varied behavior than differential equations because the points jump discontinuously along their orbits rather than flow continuously.[5]

In this paper, we have considered the model f(x) = x + μx(1 − x²) where x ∈ [0, 1] and μ ∈ [0, 2] is an adjustable parameter. Here a detailed analysis of period doubling bifurcation of model has been discussed. Here we shall also study some associated universalities, particularly the route from order to chaos, as developed by Mitchell’s J Feigenbaum an American physicist.

Secondly, we have determined the accumulation point and draw the bifurcation graph of the model and verify that chaos occur beyond accumulation point.

Thirdly, the graph of Lyapunov exponent confirms about existence of chaotic region.

Fourthly, the graphs of time series analysis are confirmed in order to support our periodic orbits of periods 2⁰, 2¹, 2², ......... [6, 7, 8]

2. Fixed point:[7, 8]

Let X be a topological space and f: X → X be a map. A real number x* is called a fixed point of the function f iff f(x*) = x*. Our model is f(x) = x + μx(1 − x²) where x ∈ [0, 1] and μ ∈ [0, 2]. Clearly the solution f(x*) = x* gives the fixed point of f. A fixed point x* is said to be

i) Stable fixed point or attractor if |f'(x*)| < 1.
ii) Unstable fixed point or repelled if |f'(x*)| > 1.
iii) Super attractive or super stable if f'(x*) = 0.

The physical significance of a ‘fixed point’ is that it can be thought as an ‘equilibrium point’. [16]

3. Periodic point and Periodic orbit

Let X be a topological space and x ∈ X. For any m ∈ Z⁺, we say x is a periodic point of period m point if fᵐ(x) = x and f(j) ≠ x for j = 1, 2, 3, ....... (m − 1). Under this circumstance the orbit of x is called a periodic orbit or period m orbit. Also we say that m is the period of the periodic orbit.

Note: A fixed point can be included under this definition so periodic points of period one. Conversely, a periodic point of period m of a map f can be viewed as the fixed point(s) of mth degree iteration of the map.

4. Bifurcation & Bifurcation point[17]

Fixed points can be created or destroyed, or their stability can be changed. These qualitative changes in the dynamic are called bifurcations and the parameter values at which they occur are called bifurcation points.

5. Chaos[10]

Chaos is the term used to describe the apparently complex behavior of what we consider to be sample, well behaved systems. Chaotic behavior, when we looked at casually, looks erratic and almost random--almost like the behavior of a system strongly influenced by outside, random ‘noise’ or complicated behavior of a system with many, many degrees of freedom, each ‘doing its own thing’
Also, some sudden and dramatic changes in nonlinear systems may give rise to the complex behavior called chaos. The noun chaos and adjective chaotic are used to describe the time behavior of a system. When that behavior is a periodic (it never exactly repeats) and is apparently random or hoist.

6. Our Study on our model

a) Finding maximum value of \( f(x) \) and its range:

In our model, at \( x = + \sqrt{\frac{1+\mu}{3\mu}} \), \( f(x) \) is maximum as \( f''(x) < 0 \) for \( \mu > 0 \) and at this point maximum value of \( f(x) \) is \( \frac{2(1+\mu)^2}{3\sqrt{3\mu}} \). Again if take \( \mu = 2 \), then we get \( f(x) \) is maximum at \( x = \frac{1}{\sqrt{2}} \) and this maximum value is 1.414 as \( x > 0 \). Hence the range may be taken as \([0, 1.414]\).

b) Finding fixed points & our interest:

Solving \( f(x) = x \), we get \( x = -1, 0, 1 \)

Therefore, the fixed points are \(-1, 0, 1\) and out of all these points our interesting points are \(0, 1\) and \(x < 0\).

c) Finding the first bifurcation point:

By stability criterion, we know that the fixed point \( x = x^* \) is stable one if \( |f'(x)| < 1 \), otherwise unstable. In case of our model, \( f'(x) = 1 + \mu - 3\mu x^2 \) and \( f''(x) \) at \( x = 0 \) is 0 and it is an unstable fixed point. Again \( f'(x) \) at \( x = 1 \) is 1 - 2\( \mu \). This absolute value remains less than 1 for \( 0 < \mu < 1 \) and greater than 1 for \( \mu > 1 \). Therefore, the fixed point \( x = 1 \) is stable for \( \mu \in (0, 1) \) and unstable otherwise. Hence \( \mu = 1 \) is the first bifurcation point of our model and their intersection point \( f(x) = x + \mu x(1 - x^2) \) and \( f(x) = x \) give the fixed points of \( f \).

(d) Finding the 2nd bifurcation point and the others:

For the second bifurcation point we consider the iterated map \( f^2(x) \) and the fixed points of it are given by solving the equation \( f^2(x) = x \), which is a ninth degree equation, so give nine roots.

With the help of C-programming and using Newton-Rapson method we get the second bifurcation point \( \mu_2 = 1.236067749978950 \) and the corresponding periodic point is \( x = 0.707107 \).

By the same way we get the other bifurcation points and also given one graph in this regard--------

7. Numerical algorithm to find periodic points, derivatives of different iterates of our map and the bifurcation points

Newton-Recurrence formula is:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ x_n = x_{n-1} + \mu x_{n-1} (1 - x_{n-1}^2) \]

Let the initial value of \( x \) be \( x_0 \). Then

\[ f(x_0) = x_0 + \mu x_0 (1 - x_0^2) = x_1 (say) \]

\[ f^2(x_0) = f(x_1) = x_1 + \mu x_1 (1 - x_1^2) = x_2 (say) \]

Proceeding in this way we get the recurrence formula as follows:

\[ x_n = x_{n-1} + \mu x_{n-1} (1 - x_{n-1}^2) \]

Now, the derivative \( f^{(k)} \) can be defined as follows:

\[ \frac{df}{dx} \bigg|_{x=x_0} = 1 + \mu - 3\mu x_0^2. \]

And by chain rule of different
ion we get
\[
\frac{d^2r}{dx^2} \bigg|_{x=x_0} = \frac{d}{dx} \left( f(x_0) \right) \frac{d}{dx} \bigg|_{x=x_0} = (1 + \mu - 3\mu x_0^2)(1 + \mu - 3\mu x_0^2).
\]
x₁ = f(x₀)

Proceeding in this way we get
\[
\frac{d^{k}f}{dx^{k}} \bigg|_{x=x_0} = (1 + \mu - 3\mu x_{k-1}^2)(1 + \mu - 3\mu x_{k-2}^2) - \ldots - (1 + \mu - 3\mu x_0^2).
\]

We always remember that the bifurcation value for the map fⁿ where its derivative \(\frac{d^{k}f}{dx^{k}}\) at periodic point is equal to −1. Now it is given below a table of the bifurcation points and one of the fixed points(periodic points) at the corresponding bifurcation point and Feigenbaum delta value is
\[
\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}
\]

### Numerical calculation of bifurcation points

<table>
<thead>
<tr>
<th>Bifurcation Points</th>
<th>One of the periodic points</th>
<th>Feigenbaum delta value (Experimental value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_1)</td>
<td>1.0000000000000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>(\mu_2)</td>
<td>1.2669679774997895</td>
<td>0.707107</td>
</tr>
<tr>
<td>(\mu_3)</td>
<td>1.2880371544828430</td>
<td>0.69182</td>
</tr>
<tr>
<td>(\mu_4)</td>
<td>1.2992727936502040</td>
<td>0.587954</td>
</tr>
<tr>
<td>(\mu_5)</td>
<td>1.3016289140370815</td>
<td>0.583623</td>
</tr>
<tr>
<td>(\mu_6)</td>
<td>1.3021432715814591</td>
<td>0.583614</td>
</tr>
<tr>
<td>(\mu_7)</td>
<td>1.3022534595730905</td>
<td>0.589555</td>
</tr>
<tr>
<td>(\mu_8)</td>
<td>1.3022770594856267</td>
<td>0.589465</td>
</tr>
<tr>
<td>(\mu_9)</td>
<td>1.3023275895437916</td>
<td>0.589471</td>
</tr>
</tbody>
</table>

From the above table we conclude that the Feigenbaum delta converge to 4.669200- - - - - and the following bifurcation diagram indicates the universal route to chaos for our model.

\[\text{Figure 7.1: Bifurcation diagram of } f(x) = x + \mu(x - x^2)\]

8. **Accumulation point: [6, 7, 8]**

Since our model follows a period doublingbifurcation, therefore we can consider that \(\{\mu_n\}\) be the sequence of bifurcation points. With the help of Feigenbaum delta (\(\delta\)), if we know first(\(\mu_1\))and second(\(\mu_2\)) bifurcation points, then we get \(\delta_3 = \frac{\mu_2 - \mu_1}{\mu_2 - \mu_3}\) .... (i) Similarly we get \(\delta_4 = \frac{\mu_3 - \mu_2}{\mu_3 - \mu_4}\) .... (ii).

From (i), (ii) we get, \(\mu_4 \approx (\mu_2 - \mu_1) \left( \frac{\delta_2}{\delta_3} \right) + \mu_2\). If we go on this procedure to calculate \(\mu_5, \mu_6\) and so on, we just obtain more terms in the sum involving powers of \(\left( \frac{\delta}{3} \right)\).

We acknowledge this sum as a geometricseries and after simplification we obtain the result. [10]
\[
\mu_n = \mu_{n+1} - \mu_{n+1} \ldots \ldots \ldots \ldots (iii)
\]

The expression (iii) is exact when the bifurcation ratio \(\delta_n = \frac{\mu_{n+1} - \mu_n}{\mu_{n+2} - \mu_{n+1}}\) is equal \(\forall n\) and then \(\lim_{n \to \infty} \delta_n = \delta\). Hence \(\{\mu_{\infty, n}\}\) is the sequence and \(\lim_{n \to \infty} \mu_{\infty, n} = \mu_\infty\). Using the experimental bifurcation points the sequence of accumulation points \(\{\mu_{\infty, n}\}\) are calculated for some values of \(n\) and the points are mentioned under in this regard.

\[
\mu_{\infty, 1} = 1.3026986339498275200266825839056
\]
\[
\mu_{\infty, 2} = 1.3020279780802696963204979703
\]
\[
\mu_{\infty, 3} = 1.30231434778964536948703371943
\]
\[
\mu_{\infty, 4} = 1.302250338560560506548353471655
\]
\[
\mu_{\infty, 5} = 1.33213273214813591727213505816297
\]
\[
\mu_{\infty, 6} = 1.302300677630961979795150636487
\]

The above sequence converge to the value 1.3023 - - - - - - which is the required accumulation point.

9. **Lyapunov Exponent**

The Lyapunov exponent is an experimental device. This device is strong and efficacious. It has ability to separate unstable, chaotic behavior from that which is stable and predictable. With the help of it we can measure these properties also. All the chaotic systems definitely have the phenomenon of sensitive dependence on initial conditions (or perturbations of the orbit). Out of all signatures one of the signatures of chaos is the divergence of adjacent (nearby) trajectories. The adjacent trajectories diverge exponentially on strange attractors. Lyapunov exponent quantifies the exponentially divergence of two trajectories starting very close to each other. This exponent has two types . . . . .

(i) The first one is positive Lyapunov exponent. It indicates the exponential divergence of the trajectory which confirms chaos.

(ii) The second is negative Lyapunov exponent. This is associated with regular behavior (periodic orbit).

[4]

The Lyapunov exponent(L) computed using the derivative method is defined by
\[
L = \lim_{n \to \infty} \frac{1}{n} \sum_{n=1}^{n} \left[ \log |f_x^n(x_1)| + \log |f_x^n(x_2)| \right] = \ldots - \ldots - \ldots - \log |f_x^n(x_n)| \right]
\]

Here \(f_x^n\) represents differentiation with respect to \(x\) and \(x_1, x_2, x_3, \ldots, x_n\) are successive iterates.[15]

In this paper, Lyapunov exponent is calculated, to verify how much accurate are the accumulation points.
From the graph of Lyapunov experiment, we see that some portions lie in the negative side of the parameter axis indicating regular behavior (periodic orbit) and the portions lie on the positive side of parameter axis confirm us about the assistance of chaos for a model.

**10. Time series analysis[5]**

It is a type of plot. It is frequently used for visualization of the solution of one dimensional difference equation of the form \( x_{n+1} = f(x_n) \), \( n=0,1,2,3,4 \ldots \) given by a map called the time series. It consists of a representation of the variable \( x_n \) as a function of \( n \). Typically the horizontal axis represents \( n \) and the vertical axis represents \( x_n \). In case of the map we have considered, the difference equation is given by \( x_{n+1} = x_n + \mu x_n (1-x_n^2) \), \( n=0,1,2,3,4 \ldots \).

The set forwards which the \( x \) values converge is called an attractor. The above figures show that an attractor may be a fixed point, a limit cycle (or a periodic attractor) or a chaotic attractor. In case of the map, we have considered, if we start with a value of \( \mu \) less than 1, successive points ‘flow’ to a fixed point at a non-zero value of \( \mu \) indicating period one behavior in Fig 10.1. However, for values of \( \mu \) slightly greater than 1 the fixed point ‘bifurcates’ to a ‘limit cycle’ of period 2 in Fig 10.2. This then bifurcates again i.e. the period double at a larger value of \( \mu \) to a limit cycle with period 4 in Fig 10.3. As \( \mu \) increase, the period continues to double at successively closer and closer value of \( \mu \) until we have chaotic behavior in Fig 10.4. In case of the map we have considered, this is illustrated in the above figures with four values of \( \mu \) (\( \mu = 0.99 \) fixed point), (\( \mu = 1.1 \) period 2 limit cycle), (\( \mu = 1.24 \), period 4 limit cycle) and (\( \mu = 1.35 \), chaos).

**References**


8. Dutta, T.K., Jain, A.Kr. and Bhattacharjee, D., “Some Aspects of Dynamical Behavior in a One Dimensional Non Linear Chaotic Map”


