Characterizations of Distributions using Moments of k-Records
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Research Article

Abstract: In this paper, we characterize some distributions using the moments of k-records associated with a sequence of i.i.d continuous random variables. The k-records are special cases of the (upper) generalized order statistics introduced by Kamps (1995). The basic idea comes from the Cauchy-Schwarz inequality.
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1. Introduction
The concept of generalized order statistics (gos) was introduced by Kamps (1995). The random variables U(r,n,m), r ≤ n are called uniform upper generalized order statistics if they possess a joint probability density function of the form

g(u1, u2, ...... un) = k(Πj=1n−1 yj) (Πj=1n−1 (1 − uj)mj) (1 − un)k−1 on the cone 0 ≤ u1 ≤ u2 ≤ ··· ≤ un < 1 of the n-dimensional Euclidean space Rn, where n≥2, k>0 and m = (m1, m2, ...., mn−1) ∈ Rn−1 are parameters such that γr = k + n + r + Σj=1n−1 mj > 0 for all r ∈ {1,2,.....,n−1} (γn = k). Consider a sequence {Xn, n ≥ 1} of independent and identically distributed (iid) random variables (r,v’s) with a common continuous distribution function (df) F. The upper generalized order statistics (ugos) based on F (or ugos associated with {Xn, n ≥ 1}) are defined by the quantile transformation X(r,n,m,k) = F−1(U(r,n,m,k)), 1 ≤ r ≤ n. When m1 = m2 = ··· = mn−1 = −1, the resulting ugos are called upper k-record values. We denote the nth upper k-record value by X(n,k).

2. Preliminaries
Lemma 2.1: If E[X] < ∞ then E[X(n,k)]δ < ∞ for all δ < γ and all n ≥ 1 where δ and γ are positive integers.
Proof: Let U(n,k) be the nth k-record associated with the standard uniform distribution. Then X(n,k) = F−1(U(n,k)). The p.d.f of U(n,k) is (Kamps, 1995)

\[ k^n \binom{n-1}{k-1} (1-x)^{n-1-k} \]

Hence

\[ E[X(n,k)]^\delta = \int_0^1 k^n \binom{n-1}{k-1} (1-x)^{(n-1-k)\delta} \]

by hypothesis and

\[ \int_0^1 \frac{1}{1-x} dx = \frac{k^n \binom{n-1}{k-1} (1-x)^{(n-1-k)\delta}}{\delta (k-y-\delta)^{(n-1-k)\delta + 1}} \]

is finite.

Lemma 2.2: Let f and g be any two square integrable continuous functions defined on (0,1). Let \( a = \int_0^1 f(x)dx \) and \( b = \int_0^1 g(x)dx \). Then

\[ |\int_0^1 f(x)g(x)dx - ab| = |(\int_0^1 f^2(x)dx - a^2)^{1/2} (\int_0^1 g^2(x)dx - b^2)^{1/2} | \]

if and only if

\[ f(x) - a = \lambda g(x) - b \]

for some real \( \lambda \).
Proof: We have

\[ \int_0^1 |f(x)g(x)dx - ab| = \int_0^1 |(f(x) - a)(g(x) - b)dx | \]

\[ \leq (\int_0^1 f^2(x)dx - a^2)^{1/2} (\int_0^1 g^2(x)dx - b^2)^{1/2} \]

Cauchy-Schwarz inequality,

\[ = (\int_0^1 f^2(x)dx - a^2)^{1/2} (\int_0^1 g^2(x)dx - b^2)^{1/2}. \]
Equality holds in this inequality if and only if \( f(x) - a = \lambda(g(x) - b) \) for some real \( \lambda \).

### 3. Characterizations

In this section, \( \{X_n, n \geq 1\} \) is a sequence of independent and identically distributed (iid) random variables (r.v’s) with a common continuous distribution function (d.f) \( F \). We assume that \( E[X_i] < \infty \) so that \( E[X_i]^p < \infty \) where \( a \) is a positive integer. \( \{X(n,k), n \geq 1\} \) is the corresponding sequence of \( n \)th \( k \)-record values. By lemma 2.1, it follows that \( E[X(n,k)] \) is a positive integer. {\( E[X(n,k)] = \sum_{i=1}^{n} \mu_i \) , \( 0 < i \leq 2 \alpha \)}

Note that \( E[X^\alpha(n,k)] = \int_0^1 \frac{g^n}{(n-1)!} U(u)^\alpha (-\log (1 - u))^{n-1} (1 - u)^{k-1} du \) \hfill (3.1)

where \( U(u) = F^{-1}(u) \).

**Theorem 3.1:**

a) If either \( k > 1, \alpha \geq 1, n \geq 2 \) or \( k = 1, \alpha \geq 2 \) (even), \( n \geq 2 \) and \( F^{-1}(0) < 0 \) then there does not exist any d.f \( F \) such that

\[ \mathbb{E} X^\alpha(n,k) - \mu_\alpha' \leq (\mu_\alpha - \mu_\alpha')^{\frac{1}{2}} \left\{ \left( \frac{2n - 2}{n} \right) \frac{k^{\alpha}}{(2k - 1)^{n-1}} - 1 \right\}^{\frac{1}{2}} \] \hfill (3.2)

b) Let \( k = 1, \alpha \geq 1, n \geq 2 \) and \( F^{-1}(0) \geq 0 \). Then (3.2) holds if and only if

\[ F(t) = 1 - \exp \left( -\left( \frac{2n - 2}{n} \right) \frac{k^{\alpha}}{(2k - 1)^{n-1}} t \right) \]

and \( F(t) \) is monotonic in \((0,1)\). Hence (3.3) cannot be satisfied for any d.f \( F \).

c) Let \( k = 1, \alpha \geq 1, n \geq 2 \) and \( F^{-1}(0) \leq 0 \). Then (3.2) holds if and only if

\[ F(t) = 1 - \exp \left( -\frac{2}{n} \frac{k^{\alpha}}{(2k - 1)^{n-1}} t \right) \]

and \( F(t) \) is monotonic in \((0,1)\). Hence (3.3) cannot be satisfied for any d.f \( F \).

**Proof:** Taking \( f(u) = \{ F^{-1}(u) \}^\alpha u \) and \( g(u) = \frac{k^n}{(n-1)!} (-\log (1 - u))^{n-1} (1 - u)^{k-1} \)

In (2.3) we notice that \( \int_0^- f(x) g(x) dx = \mathbb{E} X^\alpha(n,k) \) (from (3.1)), \( a = \mu_\alpha' \), \( b = 1 \), \n
\[ \int_0^- f^2(x) dx = \mu_\alpha^2 \] and \n
\[ \int_0^- g^2(x) dx = \int_0^- \frac{k^n}{(n-1)!} (-\log (1 - u))^{2(n-1)} (1 - u)^{2(k-1)} du. \]

The substitution \(-\log (1 - u) = t \) gives

\[ \int_0^- g^2(u) du = \int_0^- \frac{k^n}{((n-1)!)^2} e^{-(2k-1)t} t^{2(n-1)} dt = \left( \frac{2n - 2}{n} \right)^{\frac{1}{2}} \frac{k^{2n}}{(2k - 1)^{n-1}}. \]

Now (2.1) reduces to (3.2). By (2.2), (3.2) holds if and only if \( \{ F^{-1}(u) \}^\alpha = \mu_\alpha' + \lambda(g(u) + 1) \) for some \( \lambda \) real. \hfill (3.3)

a) First let \( k > 1, \alpha \geq 1 \) and \( n \geq 2 \). Since \( k > 1 \), \( g(u) \) is increasing in the interval \((0, u_0)\) and decreasing in the interval \((u_0, 1)\) where \( u_0 = 1 - e^{\frac{1}{2k-1}} \).

Hence the right side of (3.3) is not monotonic in \((0,1)\). But, the left side of (3.3) is non-increasing in \((0,1)\). Hence (3.3) cannot be satisfied for any d.f \( F \).

Now let \( k = 1, \alpha \geq 2 \) even, \( n \geq 2 \) and \( F^{-1}(0) < 0 \). Note that \( g(u) = \frac{-\log (1 - u)}{(n-1)!} \), \( u > 0 \), \( 0 < u < 1 \). Since the left side of (3.3) is non-decreasing, we must have \( \lambda > 0 \). Hence the right side of (3.3) is negative for

\[ 0 < u < u_1 = 1 - \exp \left( -\left( \frac{2}{n} \frac{k^{\alpha}}{(2k - 1)^{n-1}} \right)^{\frac{1}{n-1}} \right) \].

But, the left side of (3.3) is positive since \( \alpha \geq 2 \) is even. Thus (3.3) cannot be satisfied for any d.f \( F \).

b) Since \( k = 1 \), we have \( g(u) > 0 \), \( 0 < u < 1 \). Since the left side of (3.3) is non-decreasing, we must have \( \lambda > 0 \). Since \( F^{-1}(0) \geq 0 \) and \( g(0) = 0 \), it follows that \( \mu_\alpha - \lambda \geq 0 \) and \( F^{-1}(u) = \{ \mu_\alpha + \lambda(g(u) - 1) \}^\frac{1}{n-1}, 0 < u < 1 \)

Note that \( g'(x) = 1 - \exp \left( -\left( x(n-1)! \right)^{\frac{1}{n-1}} \right), x > 0 \).

Putting \( t = (\mu_\alpha + \lambda g(u) - 1)^{\frac{1}{n-1}} \), we notice that \( t > (\mu_\alpha - \frac{1}{n-1})^{\frac{1}{n-1}} \geq 0 \) and

\[ F(t) = g^{-1} (t^{\frac{1}{n-1}} + \lambda - \mu_\alpha) \]
Also, \( a = (i) \) and \( (ii) \). Hence there does not exist any d.f \( F \) for which \( \lambda \).

Proceeding as in case b) , we get

\[
F(t) = 1 - \exp\left\{-\left[\frac{(n-1)(\lambda - \mu_{\alpha})}{\lambda}\right]^{\frac{1}{n-1}}\right\}, \quad t > (\mu_{\alpha} - \lambda)^{\frac{1}{n-1}}, \quad \lambda > 0
\]

\( c) \) Let \( k = 1 \), \( \alpha (\geq 1) \) odd, \( n \geq 2 \) and \( F' (0) < 0 \). As in case b) , we have \( \lambda > 0 \).

\[
F^{-1}(u) < 0 \text{ for } 0 < u < u_1 \text{ and } F'(u) > 0 \text{ for } u_1 < u < 1, \text{ where } u_1 \text{ is as in case a).}
\]

Let \( 0 < u < u_1 \), Then \( F^{-1}(u) = \text{h}(u), \text{ where } \text{h}(u) > 0. \text{ From (3.3) , we have } (-\text{h}(u))^\alpha = \mu_{\alpha} + \lambda \text{ (g}(u) - 1) \text{. This gives}
\]

\[
F^{-1}(u) = \left(-\left(\mu_{\alpha} + \lambda (g(u) - 1))\right)^\frac{1}{\alpha}, \quad 0 < u < u_1.
\]

Proceeding as in case b) , we get

\[
F(t) = 1 - \exp\left\{-\left[\frac{(n-1)(\lambda - \mu_{\alpha}+^\alpha)}{\lambda}\right]^{\frac{1}{n-1}}\right\}, \quad -\lambda < \mu_{\alpha} < t < 0 .
\]

Now let \( u_1 < u < 1 \). In this case \( F^{-1}(u) = \left[\mu_{\alpha} + \lambda (g(u) - 1))\right]^\frac{1}{\alpha}, u_1 < u < 1. \)

Proceeding as in case b) , we get

\[
F(t) = 1 - \exp\left\{-\left[\frac{(n-1)(\lambda - \mu_{\alpha}+^\alpha)}{\lambda}\right]^{\frac{1}{n-1}}\right\}, \quad t > 0.
\]

Remark: 3.1: In case b) , the d.f is Weibull distribution with shifted origin.

Theorem 3.2: Let \( F \) be a continuous d.f symmetric about the origin. Let \( k \), \( \alpha \) and \( n \) be integers such that

(i) \( k > 1, \alpha \geq 1, n \geq 2 \) or

(ii) \( k = 1, \alpha \geq 2 \text{(even)}, n \geq 2 \) or

(iii) \( k = 1, \alpha (\geq 1) \) odd, \( n = 2. \)

\[
a) \quad \text{In cases (i) and (ii) there does not exist any d.f } F \text{ such that}
\]

\[
\left|EX^\alpha(n, k) - b\mu_{\alpha} = (\mu_{\alpha}^\alpha - \mu_{\alpha}^2)\frac{1}{2}\left(\frac{k^n}{2(2(n-1)!)}(\log(1-u))^n-1(1-u)^{k-1}(\log u)^{n-1}u^{k-1}du\right)
\]

\[
\text{where } b = \frac{1}{(n-1)!}2\int_0^1(1-u)^{n-1}(1-u)^{k-1}(\log u)^{n-1}u^{k-1}du.
\]

b) In case (iii), \( F(t) = 1 - \exp\left\{-\left[\frac{(n-1)(\lambda - \mu_{\alpha}+^\alpha)}{\lambda}\right]^{\frac{1}{n-1}}\right\}, \quad -\infty < t < \infty, \lambda > 0. \)

Proof: Take \( f(u) = \{F^{-1}(u))\}, \text{ and } \]

\[
g(u) = \frac{kn}{2(n-1)!}(\log(1-u))^n-1(1-u)^{k-1}(-1)^n(-\log u)^n-1u^{k-1}
\]

\[
\text{In (2.1). Then } \int_0^1 f(u)g(u)du = \frac{k^n}{2(n-1)!}\int_0^1 u^n(\log(1-u))^n-1(1-u)^{k-1}du
\]

\[
\frac{k^n}{2(n-1)!}\int_0^1 u^n(-\log u)^n-1u^{k-1}du.
\]

\[
\frac{1}{2}EX^\alpha(n, k) + \frac{k^n}{2(n-1)!}\int_0^1 U(1-u)^{\alpha}(-\log u)^n-1u^{k-1}du.
\]

( from (3.1) and the fact that the symmetry of F about zero implies)

\[
U(u) = -\{I - \text{d}(.), 0 < \text{d} < 1. \}
\]

\[
= \frac{1}{2}\int_0^1\{\log(1-u)\}(-\log(I - \text{d}))^{n-1}(I - \text{d})^\alpha = \text{d}(\text{d}).
\]

Also, \( a = \text{d}', b = \frac{\alpha(-\text{d})}{2}, \int_0^1\text{d}^2(\text{d})\text{d} = \text{d}^2 \), and

\[
\int_0^1\text{d}^2(\text{d})\text{d} = \frac{\text{d}^2}{2}\left(\frac{2(2(n-1)!)}{(\log u)^{n-1}}(-1)^\text{d}(\text{d})\text{d} \right), \text{ where } A_{n,k} \text{ is as in (a).}
\]

Then (2.1) reduces to (3.4). By (2.2), (3.4) holds good if and only if

\[
\{U^{-1}(\text{d})\}^\alpha - \text{d} = \text{d}(\text{d}) - \text{d} \text{ for some } \lambda > 0.
\]

(3.5)

a) Note that the left side of (3.5) is non-decreasing in \( (0,1) \) whereas then right side is not non-decreasing in \( (0,1) \) in cases (i) and (ii). Hence there does not exist any d.f \( F \) for which (3.4) is true.

b) Let \( k = 1, n = 2 \) and \( \alpha \geq 1 \) (odd). Then (3.5) reduces to

\[
\{U^{-1}(\text{d})\}^\alpha - \text{d} = \frac{1}{\alpha}\log\left(\frac{1}{\text{d}}\right), \quad 0 < \text{d} < 1.
\]
Since \((F^{-1}(u))^\alpha\) is non-decreasing, it follows that \(\lambda > 0\). Since \(\alpha\) is odd, we see that \(F^{-1}(u)\) is negative or positive according as \(0 < u < \frac{1}{2}\) or \(\frac{1}{2} < u < 1\).

First let \(0 < u < \frac{1}{2}\). Then

\[
-\frac{1}{l} = -\left(\frac{l}{2}\right) \left(\frac{\log(1-u)}{\log(1-u)}\right)^{\frac{1}{\lambda}}.
\]

Putting \(\theta = -\left(\frac{l}{2}\right) \left(\frac{\log(1-u)}{\log(1-u)}\right)^{\frac{1}{\lambda}}\) and solving for \(u\), we get

\[
\theta = \frac{l}{l + \frac{\lambda}{\theta}} = \frac{l}{l + \frac{\lambda}{\theta}}, \quad \theta < 0.
\]

Similarly for \(\frac{1}{2} < u < 1\), we get

\[
\theta = \frac{l}{l + \frac{\lambda}{\theta}} = \frac{l}{l + \frac{\lambda}{\theta}}, \quad \theta > 0.
\]

Since \(\alpha\) is odd, we observe that

\[
\theta = \frac{l}{l + \frac{\lambda}{\theta}}, \quad -\infty < \theta < \infty, \quad \theta > 0.
\]

**Remark 3.2:** Note that in case b) \(F\) reduces to the logistic d.f when \(\alpha = 1\).

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