

Explicit Bounds on Certain Integral Inequalities and its Application

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Research Article

Abstract: In this paper, we establish some new retarded integral inequalities in two variables. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of differential equation. Some applications are also given.

Keywords: Explicit bound, two independent variable, integral inequality, partial derivatives, boundedness.

Introduction

In the development of the theory of differential and integral equations integral inequalities which provide explicit bounds on unknown functions take very

important place. For instant the explicit bounds given by the well-known Gronwall-Bellman [1,3] inequality and its nonlinear generalization due to Bihari [2] are used to considerable extent in the literature [1-12]. The main purpose of this paper is to establish explicit bounds on retarded Gronwall-Bellman and Bihari-like inequalities in two variables which can be used to study the qualitative behavior of the solutions of certain classes of retarded partial differential equations. Some applications of one of our result are also given.

Main Results

In what follows, R denotes the set of real numbers; $R_+ = [0, \infty)$, $R^* = (0, \infty)$, $R_1 = [1, \infty)$, $J_1 = [x_0, X)$, and $J_2 = (y_0, Y)$ are the given subsets of R, $\Delta = J_1 \times J_2$. The first order partial derivatives of a function $z(x,y)$ with respect to x and y are denoted by $D_1 z(x,y)$ and $D_2 z(x,y)$, respectively.

Theorem 2.1: Let $u, g, h \in C(\Delta, R_+)$ and $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ be non decreasing with $C^1(J_2, J_2)$ be nondecreasing with $\alpha(x) \leq x$ on J_1 , $\beta(y) \leq y$ on J_2 . Let $f(x,y)$ be nondecreasing in $(x,y) \in \Delta$. If the inequality.

$$u(x,y) \leq f(x,y) + \int_{x_0}^x \int_{y_0}^y g(s,t) u(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h(s,t) u(s,t) dt ds \quad (2.1)$$

holds, then

$$u(x,y) \leq f(x,y) \exp[G(x,y) + H(x,y)] \quad (2.2)$$

for $(x,y) \in \Delta$, where

$$G(x,y) = \int_{x_0}^x \int_{y_0}^y g(s,t) dt ds \quad (2.3)$$

$$H(x,y) = \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h(s,t) dt ds \quad (2.4)$$

Proof : Since $f(x,y)$ is positive and nondecreasing, we can restate (2.1) as

$$\frac{u(x,y)}{f(x,y)} \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) \frac{u(s,t)}{f(s,t)} dt ds + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h(s,t) \frac{u(s,t)}{f(s,t)} dt ds \quad (2.5)$$

Let $r(x,y) = \frac{u(x,y)}{f(x,y)}$ then

$$r(x,y) \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) r(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} h(s,t) r(s,t) dt ds \quad (2.6)$$

Define a function $z(x,y)$ by the right - hand side of (2.6), then we have

$$z(x,y) = 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) r(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) r(s,t) dt ds \quad (2.7)$$

Then it is clear that

$$r(x,y) \leq z(x,y), z(x_0,y) = z(x,y_0) = 1 \quad (2.8)$$

Differentiate (2.7) with respect to x, we get

$$D_1 z(x,y) = \int_{y_0}^y g(x,t) r(x,t) dt + \int_{\beta(y_1)}^{\beta(y)} h(\alpha(x),t) r(\alpha(x),t) dt \alpha'(x)$$

Using (2.8), we have

$$\begin{aligned} & \leq \int_{y_0}^y g(x,t) z(x,t) dt + \int_{\beta(y_1)}^{\beta(y)} h(\alpha(x),t) z(\alpha(x),t) dt \alpha'(x) \\ & \leq z(x,y) \left(\int_{y_0}^y g(x,t) dt + \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) dt \alpha'(x) \right) \end{aligned} \quad (2.9)$$

$$\frac{D_1 z(x,y)}{z(x,y)} \leq \int_{y_0}^y g(x,t) dt + \int_{\beta(y_1)}^{\beta(y)} h(\alpha(x),t) dt \alpha'(x) \quad (2.10)$$

Keeping y fixed in (2.10), setting x = σ and integrating it with respect to σ from x_0 to x, x ∈ J_1 and making change of variable, we get

$$z(x,y) \leq \exp [G(x,y) + H(x,y)]$$

for (x,y) ∈ Δ

Using (2.8), we have

$$r(x,y) \leq \exp [G(x,y) + H(x,y)]$$

Hence

$$u(x,y) \leq f(x,y) \exp [G(x,y) + H(x,y)]$$

Theorem 2.2 : Let u,g,h ∈ C(Δ, R_+) and α ∈ C^1(J_1, J_1), β ∈ C^1(J_2, J_2) be nondecreasing with α(x) ≤ x on J_1, β(y) ≤ y on J_2. Let f(x,y) be nondecreasing in (x,y) ∈ Δ and p > 1 is a constant. If the inequality

$$u^p(x,y) \leq f^p(x,y) + \int_{x_0}^x \int_{y_0}^y g(s,t) u(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) u(s,t) dt ds \quad (2.11)$$

holds, then

$$u(x,y) \leq f(x,y) \left[1 + \left(\frac{p-1}{p} \right) [Q(x,y) + w(x,y)] \right]^{\frac{1}{p-1}} \quad (2.12)$$

where

$$Q(x,y) = \int_{x_0}^x \int_{y_0}^y f^{1-p}(s,t) g(s,t) dt ds \quad (2.13)$$

and

$$W(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f^{1-p}(s,t) h(s,t) dt ds \quad (2.14)$$

Proof : Since f(x,y) is positive and non decreasing in both variables equation (2.11) rewrite as

$$\frac{u^p(x,y)}{f^p(x,y)} \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-p}(s,t) \frac{u(s,t)}{f(s,t)} dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-p}(s,t) \frac{u(s,t)}{f(s,t)} dt ds \quad (2.15)$$

Let $r(x,y) = \frac{u(x,y)}{f(x,y)}$ then

$$r^P(x,y) \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-P}(s,t) r(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-P}(s,t) r(s,t) dt ds \quad (2.16)$$

Define a function $z(x,y)$ by the right hand side of (2.16) then we have

$$z(x,y) = 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-P}(s,t) r(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-P}(s,t) r(s,t) dt ds \quad (2.17)$$

then it is clear that

$$r^P(x,y) \leq z(x,y), z(x_0,y) = z(x,y_0) = 1 \quad (2.18)$$

Differentiation (2.17) with respect to x , we have

$$D_1 z(x,y) = \int_{y_0}^y g(x,t) f^{1-P}(x,t) r(x,t) dt + \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) f^{1-P}(x,t) r(x,t) dt \cdot \alpha'(x)$$

Using (2.18) we get

$$\begin{aligned} D_1 z(x,y) &\leq \int_{y_0}^y g(x,t) f^{1-P}(x,t) z^{\frac{1}{P}}(x,t) dt + \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) f^{1-P}(x,t) z^{\frac{1}{P}}(x,t) dt \cdot \alpha'(x) \\ \frac{D_1 z(x,y)}{z^{\frac{1}{P}}(x,y)} &\leq \int_{y_0}^y g(x,t) f^{1-P}(x,t) dt + \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) f^{1-P}(\alpha(x),dt) \cdot \alpha'(x) \end{aligned} \quad (2.19)$$

Keeping y fixed in (2.19), setting $x = \sigma$ and integrating it with respect to σ from x_0 to x , $x \in J_1$ and making change of variable, we get

$$\begin{aligned} \frac{p}{p-1} z^{\frac{p-1}{p}}(x,y) &\leq \frac{p}{p-1} + \int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-P}(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-P}(s,t) dt ds \\ z^{\frac{p-1}{p}}(x,y) &\leq 1 + \frac{p-1}{p} \left[\int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-P}(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-P}(s,t) dt ds \right] \end{aligned} \quad (2.20)$$

Using (2.18) in (2.20) we get

$$\begin{aligned} r^{P-1}(x,y) &\leq 1 + \frac{p-1}{p} \left[\int_{x_0}^x \int_{y_0}^y g(s,t) f^{1-P}(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) f^{1-P}(s,t) dt ds \right] \\ r^{P-1}(x,y) &\leq 1 + \frac{p-1}{p} [Q(x,y) + W(x,y)] \end{aligned}$$

where $Q(x,y)$ and $W(x,y)$ are mentioned in (2.13) & (2.14).

$$\text{Hence } u(x,y) \leq f(x,y) \left[1 + \frac{p-1}{p} [Q(x,y) + W(x,y)] \right]^{\frac{1}{p-1}} \quad (2.21)$$

Theorem 2-3: Let $u, g, h \in C(\Delta, R_+)$, $f \in C(\Delta, R_+^*)$ and $\alpha \in C^1(J_1, J_1)$, $\beta \in C^1(J_2, J_2)$ be non-decreasing with $\alpha(x) \leq x$ on J_1 , $\beta(y) \leq y$ on J_2 . For $i = 1, 2$, let $g_i \in C(R_+, R_+)$ be non-decreasing function with $g_i(u) > 0$ for $u > 0$ and $\frac{g_i(u(t))}{f(t)} \leq g_i\left(\frac{u(t)}{f(t)}\right)$.

If the inequality

$$u(x,y) \leq f(x,y) + \int_{x_0}^x \int_{y_0}^y g(s,t) g_1(u(s,t)) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) g_2(u(s,t)) dt ds \quad (2.22)$$

for $(x,y) \in \Delta$,

(i) In case $g_1(u) \leq g_2(u)$

$$u(x,y) \leq f(x,y) G_2^{-1} [G_2(1) + G(x,y) + H(x,y)] \quad (2.23)$$

(ii) In case $g_2(u) \leq g_1(u)$

$$u(x,y) \leq f(x,y) G_1^{-1} [G_1(1) + G(x,y) + H(x,y)] \quad (2.24)$$

Where $G(x,y)$ and $H(x,y)$ are defined by (2.3) and (2.4) and for $i = 1, 2$ G_i^{-1} are the inverse functions of

$$G_i(r) = \int_{r_0}^r \frac{ds}{g_i(s)}, r > 0, r_0 > 0 \quad (2.25)$$

and $x \in J_1, y \in J_2$ are so chosen that for $i = 1, 2$

$$G_i(1) + G(x,y) + H(x,y) \in \text{Dom}(G_i^{-1})$$

respectively, for all x, y lying in $[x_0, x_1]$ and $[y_0, y_1]$.

Proof : Since $f(x,y)$ is positive and non-decreasing in both variables (2.22) Can be rewrite as

$$\frac{u(x,y)}{f(x,y)} \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) g_1 \left(\frac{u(s,t)}{f(s,t)} \right) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) g_2 \left(\frac{u(s,t)}{f(s,t)} \right) dt ds \quad (2.26)$$

$$\frac{u(x,y)}{f(x,y)} \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) g_1 \left(\frac{u(s,t)}{f(s,t)} \right) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) g_2 \left(\frac{u(s,t)}{f(s,t)} \right) dt ds \quad (2.27)$$

$$\text{Let } r(x,y) = \frac{u(x,y)}{f(x,y)}$$

Hence we obtain

$$r(s,t) \leq 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) g_1(r(s,t)) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) g_2(r(s,t)) dt ds \quad (2.28)$$

Define $z(x,y)$ by the right hand side of (2.28), we get

$$z(x,y) = 1 + \int_{x_0}^x \int_{y_0}^y g(s,t) g_1(r(s,t)) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) g_2(r(s,t)) dt ds \quad (2.29)$$

From (2.28), we have

$$r(s_1t) \leq z(x,t), z(x,y_0) = z(x_0,y) = 1 \quad (2.30)$$

Now

$$D_1 z(x,y) = \int_{y_0}^y g(x,t) g_1(u(x,t)) dt + \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) g_2(u(\alpha(x),t)) dt \alpha'(x) \quad (2.31)$$

from (2.31) in (2.30), we get

$$D_1 z(x,y) \leq \int_{y_0}^y g(x,y) g_1(z(x,t)) dt + \left(\int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) g_2(z(\alpha(x),t)) dt \right) \alpha'(x) \quad (2.32)$$

$$D_1(z(x,y)) \leq g_1(z(x,y)) \int_{y_0}^y g(x,t) dt + g_2(z(x,y)) \int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) g_2(z(\alpha(x),t)) dt \alpha'(x) \quad (2.33)$$

When $g_2(u) \leq g_1(u)$, then from (2.33), we observed that

$$\frac{D_1(z(x,y))}{g_1(z(x,y))} \leq \int_{y_0}^y g(x,t) dt + \left(\int_{\beta(y_0)}^{\beta(y)} h(\alpha(x),t) dt \right) \alpha'(x) \quad (2.34)$$

Using (2.25) in (2.34) , we get

$$D_1 G_1(z(x,y)) \leq \int_{y_0}^y g(x,t) dt + \left(\int_{\beta(y)}^{\beta(y)} h(\alpha(x),t) dt \right) \alpha'(x) \quad (2.35)$$

Keeping y fixed in (2.35) Setting x = σ, then integrating with respect to σ from x₀ to x, x ∈ J₁ and making change of variable, we get

$$G_1(z(x,y)) \leq G_1(1) + \int_{x_0}^x \int_{y_0}^y g(s,t) dt ds + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} h(s,t) dt ds \quad (2.36)$$

Using the bound on z(x,y) from (2.36) in (2.30), we get required inequality in (2.23).

Applications:

In this section we present applications of the inequality in Theorem 1 to study the boundedness and uniqueness of the solutions of the initial boundary value problem for hyperbolic partial differential equations of the form

$$D_2 D_1 u(x,y) = F(x,y, u(x,y), u(x-h_1(x), y-h_2(y))), \quad (3.1)$$

$$u(x,y_0) = e_1(x), u(x_0,y) = e_2(y), e_1(x_0) = e_2(y_0) = 0 \quad (3.2)$$

Where F ∈ C(Δ × R², R), e₁ ∈ C¹(J₁, R), e₂ ∈ C¹(J₂, R), h₁ ∈ C¹(J₁, R+), h₂ ∈ C¹(J₂, R₊) such that x-h₁(x) ≥ 0, y₁ - h₂(y) ≥ 0, h₁'(x) < 1, h₂'(x) < 1 and h₁(x₀) = h₂(y₀) = 0.

Our first result gives the bound on the solution of the problem (3.1) - (3.2)

Theorem 3.1: suppose that

$$|F(x,y,u,v)| \leq g(x,y) |u| + k(x,y) |v| \quad (3.3)$$

$$|e_1(x) + e_2(y)| \leq p(x,y) \quad (3.4)$$

Where g, k ∈ C(Δ, R+) and k ≥ 0 and let

$$M_1 = \max_{x \in J_1} \frac{1}{1 - h_1'(x)}, M_2 = \max_{x \in J_2} \frac{1}{1 - h_2'(y)} \quad (3.5)$$

If u(x,y) is any solution of (3.1) - (3.2) then

$$|u(x,y)| \leq p(x,y) \exp(G(x,y) + \bar{K}(x,y)) \quad (3.6)$$

where G(x,y) is defined by (2.3) and

$$\bar{K}(x,y) = M_1 M_2 \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \bar{k}(\sigma, \tau) d\sigma d\tau \quad (3.7)$$

and

$$\bar{k}(\sigma, \tau) = k(\sigma + h_1(s), \sigma + h_2(s)),$$

for σ ∈ J₁, τ ∈ J₂.

Proof : It is easy to see that the solution u(x,y) of the problem (3.1) - (3.2) satisfies the equivalent integral equation

$$u(x,y) = e_1(x) + e_2(y) + \int_{x_0}^x \int_{y_0}^y F(s,t, u(x,y), u(x-h_1(x), y-h_2(y))) dt ds \quad (3.8)$$

Using (3.3), (3.4),(3.5) in (3.8) and making change of variables , we get

$$|u(x,y)| \leq p(x,y) + \int_{x_0}^x \int_{y_0}^y g(x,y) |u(x,t)| dt ds + M_1 M_2 \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \bar{k}(\sigma, \tau) |u(\sigma, \tau)| d\sigma d\tau \quad (3.9)$$

Now suitable application of the inequality in theorem 2.1 to 3.9 yields (3.6). The right-hand side of (3.6) gives us the bound on solution u(x,y) of (3.1) - (3.2) in terms of known function. Thus if the right-hand side of (3.6) is bounded, then we assent that the solution of (3.1)-(3.2) is bounded.

The next result deals with the uniqueness of the solution the problem (3.1) - (3.2)

Theorem 3.2: Suppose that the function F in (3.1) satisfies the condition

$$|F(x,y,u,v) - F(x,y, \bar{u}, \bar{v})| \leq g(x,y) |\bar{u} - u| + k(x,y) |\bar{v} - v| \quad (3.10)$$

g,k ∈ C(Δ,R) , and let M₁, M₂, α, β , \bar{k} be as in theorem 3.1 then for problem (3.1) – (3.2) has at most one solution.

Proof : Let u (x,y) and \bar{u} (x,y) be two solution of (3.1) – (3.2) on Δ. Then we have

$$u(x,y) - \bar{u}(x,y) = \int_{x_0}^x \int_{y_0}^y [F(s,t,u(s,t)u(s-h_1(s),t-h_2(t))) - F(s,t,\bar{u}(s,t)\bar{u}(s-h_1(s),t-h_2(t)))] ds dt \quad (3.11)$$

Using (3.10) in (3.11) and making change of variables we get

$$|u(x,y) - \bar{u}(x,y)| \leq \int_{x_0}^x \int_{y_0}^y g(s,t) |z(s,t) - \bar{z}(s,t)| + M_1 M_2 \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} k(\sigma, \tau) |z(\sigma, \tau) - \bar{z}(\sigma, \tau)| d\tau d\sigma$$

Now suitable application of inequality in theorem (2.1) yields

$$|u(x,y) - \bar{u}(x,y)| \leq 0$$

Therefore $u(x,y) = \bar{u}(x,y)$ i.e there is almost one solution of problem (3.1) - (3.2).

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