A New Result on Maximum Principles

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Research Article

Abstract: In this paper we prove a maximum principle for n^{th} derivative of the function which is n times continuously differentiable on the closed interval of the real line under initial conditions. Taylor's theorem plays a vital role in the proof of our results.

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Introduction

Maximum principle plays a very important role in the study of differential equations. It gives useful information about the uniqueness, boundedness and symmetry of the solution. Maximum principles for differential equations have been discussed by Protter and Weinberger [1], R. P. Sperb [2] and in [3], [4], [5], [6]. But their discussion does not include the maximum principles for higher order derivatives. The purpose of this paper is to discuss maximum principle for higher order derivatives of a function. Let R denotes the real line and a, b $\in$ R and f is a real valued function defined on [a, b]. Maximum principle for the function gives maximum value of the function which is on the boundary of its domain or inside the domain depending on the function, domain of the function and derivative of the function. The precise definition of maximum principle is given below:

Definition

We say that the maximum principle holds for f(x) on [a, b] if f(x) $\leq$ max {f (a), f (b)} on [a, b]

We know that if $f'(x) \geq 0$ or $f''(x) \geq 0$ then maximum principle holds for f(x) on [a, b]. But if $f^{(n)}(x) \geq 0$ for n $\geq$3 then maximum principle does not necessarily hold on [a, b]. For example, consider $f(x) = \pm 2x^2$ and its third derivative. We present the conditions under which $f^{(n)}$(x) $\geq$ 0 where n $\geq$ 3 does imply the maximum principle.

Let C^n(I) where I = [a, b] be the set of all real valued functions on I that are n times continuously differentiable. We know that

$$P_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i\,!}(x-a)^i$$

Is the n^{th} degree Taylor's polynomial of f (x) at a, where f (x)$\in C^n$(I) and a $\in$ I.

Main Results

We prove the following theorem which gives the basic result for maximum principle for a function f on [a, b]

Theorem1.1: Let f (x) $\in C^n$([a, b]) for some n $\geq$2.

If $f^{(n)} \geq$ 0 on [a, b] then f (x) $\leq u_n$ (x)

on [a , b] where u_n (x) = P_{n-2}(x) + $\frac{f(b)-P_{n-2}(b)}{(b-a)^{n-1}}(x-a)^{n-1}$

Proof: Let f (x)$\in C^n$([a, b]) for some n $\geq$2.

If $f^{(n)}(x) \geq 0$ on [a, b]. We have by Taylor's theorem

F (x) – P_{n-2}(x) = h(x)$(x-a)^{n-1}$

This gives

$$h(x) = \frac{f(x)-P_{n-2}(x)}{(x-a)^{n-1}} = \frac{1}{(n-2)!}\int_0^1 (1-t)^{n-2} f^{(n-1)}\big(a+t(x-a)\big)dt$$

It follows that

$$h'(x) = \frac{1}{(n-2)!}\int_0^1 t(1-t)^{n-2} f^{(n)}\big(a+t(x-a)\big)dt$$

Since $f^{(n)}(x) \geq 0$ on [a, b], therefore $h'(x) \geq 0$ on [a, b]

So h(x) $\leq$h(b) which gives $\frac{f(x)-p_{n-2}(x)}{(x-a)^{n-1}} \leq h(b)$

f (x) $\leq h(b)(x-a)^{n-1} + P_{n-2}(x) = P_{n-2}(x) + \frac{f(b)-p_{n-2}(b)}{(b-a)^{n-1}}(x-a)^{n-1}$

i.e. f(x) $\leq u_n(x)$ on [a, b]

Now we prove the following theorem which shows that maximum principle holds under initial conditions.

Theorem 2.2: let f (x)$\in C^n$([a, b]) for some n $\geq$3, if

1. $f^n(x) \geq 0$ on [a, b]
2. $f(b) \geq P_i(b)$ for i=1,2,--------,n-2

$then\, f(x) \leq \max\{f(a), f(b)\}$ on [a, b]

Proof: We prove the theorem by mathematical induction.

First we prove the theorem for the initial case n=3. Suppose that for some f (x)$\in C^3$([a, b]) , we have f''' (x) $\geq$0 on[a, b] and f(b) $\geq P_1(b)$

Then by Theorem 2.1, we have

$f(x) \leq u_3(x) = P_1(x) + \frac{f(b)-P_1(b)}{(b-a)^2}(x-a)^2$ on [a, b]

Since $P_1(x) = f(a) + f'(a)(x-a)$

$P_1'(x) = f'(a)$

$P_1''(x) = 0$

And $f(b) - P_1(b) = f(b) - [f(a) + f'(a)(b-a)] \geq 0$

So we have $u_3''(x) = P_1''(x) + \frac{f(b)-P_1(b)}{(b-a)^2}\, 2 \geq 0$

Since $P_1''(x) = 0$, f (b) $\geq P_1(b)$

Thus $u_3''(x) \geq 0$ which implies that

$u_3(x) \leq \max\{u_3(a), u_3(b)\}$ on [a, b].

It follows that $u_3(a) = f(a)$ and $u_3(b) = f(b)$. So $f(x) \leq u_3(x) \leq \max\{f(a), f(b)\}$ on [a, b].

Therefore the theorem holds for n=3.

Assume that the theorem holds in the case n=k where k≥ 3 then we show that theorem hold in the case n=k+1. To show this suppose that for some

$f(x) \in C^{(k+1)}([a,b])$

We have $f(x)^{(k+1)} \geq 0$ on [a, b] and $f(b) \geq P_i(b)$ for i =1, 2, ------, k-1.

By Theorem 2.1, we have

$f(x) \leq u_{k+1}(x) = P_{k-1}(x) + \frac{f(b)-P_{k-1}(b)}{(b-a)^k}(x-a)^k$ on [a, b]

As $P_{k-1}^{(k)}(x) = 0$ and $f(b) - P_{k-1}(b) \geq 0$, we have

$u_{k+1}^{(k)}(x) = P_{k-1}^{(k)}(x) + \frac{f(b) - P_{k-1}(b)}{(b-a)^k} k! \geq 0$, since $f(b) - P_{k-1}(b) \geq 0$

It is clear that,

$u_{k+1}^{(i)}(x) = f^{i}(a)$ for i=0,1,2,-------,k-1 and $u_{k+1}(b) = f(b)$

So $u_{k+1}(b) = f(b) \geq P_j(b) = \sum_{i=0}^{j} \frac{f^{(i)}(a)}{i!}(b-a)^i = \sum_{i=0}^{j} \frac{u_{k+1}^{(i)}(a)}{i!}(b-a)^i$ for j =1, 2, -----, k-2

Therefore by the induction assumption

$u_{k+1}(x) \leq \max\{u_{k+1}(a), u_{k+1}(b)\} = \max\{f(a), f(b)\}$ on [a, b]

So $f(x) \leq u_{k+1}(x) \leq \max\{f(a), f(b)\}$ on [a, b]

Thus Theorem holds for n=k+1.

Hence we conclude that theorem holds for any n.

Example 2.1: consider the function

f (x) = (x+2) (x+3) (x-2) (x-3)

That is $f(x) = x^4 - 13x^2 + 36$

It is not true that $f'(x)$, $f''(x) \geq 0$ on [0, 4],

and it is true that $f'''(x) \geq 0$ on [0,4]

Also f (4) =84 and $P_1(4) = 36$

Thus $f(4) \geq P_1(4)$

Therefore by Theorem 2.2 maximum principle holds for f(x) on [0, 4].

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