Bayesian Analysis for the Generalized Rayleigh Distribution

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Research Article

Abstract: In this paper, the estimation problem of generalized Rayleigh distribution is considered. The parameters are estimated using likelihood based inferential procedure: classical as well as Bayesian. We have computed MLEs and Bayes estimates under gamma priors along with their asymptotic confidence, bootstrap and HPD intervals. The Bayesian estimates of the parameters of generalized Rayleigh distribution are obtained using Markov chain Monte Carlo (MCMC) simulation method. We have obtained the probability intervals for parameters, hazard and reliability functions. The posterior predictive check method has been applied for evaluating the model fit. We have also discussed the Bayesian estimation and prediction for Type-II censored data. All the computations are performed in OpenBUGS and R software. A real data set is analyzed for illustration of the proposed inferential procedures.

Keywords: Generalized Rayleigh distribution, Markov chain Monte Carlo, Bayesian estimation, Bootstrap, OpenBUGS.

1. Introduction

The Rayleigh distribution is a special case of the Weibull distribution, which provides a population model useful in several areas of statistics, including life testing and reliability which age with time as its failure rate is a linear function of time. Rayleigh has a linearly increasing failure rate which makes it a suitable model for the lifetime of components that age rapidly with time.

In recent years, new classes of models have been proposed based on modifications of the existing model. Several exponentiated distributions have been studied quite extensively, since the work of Mudholkar and Srivastava (1993) on exponentiated Weibull distribution. The exponentiated form of exponential distribution has been introduced by Gupta and Kundu (1999) and named it as generalized exponential distribution. Along the same line of the generalized exponential distribution Surles and Padgett (2005) introduced two-parameter Burr Type X distribution and named as the generalized Rayleigh distribution, Kundu and Raqab (2005). Nadarajah and Kotz (2006) proposed several exponentiated type distributions extending the Frchet, gamma, Gumbel and Weibull distributions. The two-parameter generalized Rayleigh distribution is a member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava (1993). Kundu and Raqab (2005) and Raqab and Kundu (2006) have discussed the different methods of estimation of the parameters and other properties of generalized Rayleigh distribution. Raqab and Madi (2009) studied exponentiated Rayleigh distribution in Bayesian framework. In fact, there exist a lot of density functions which may be considered as generalizations of the Rayleigh distribution.

Recently, two versions of generalized Rayleigh distribution (GRD) are appeared in literature Voda (2007, 2009). The probability density function of first one has the following form :

$$f(x;\alpha,\lambda)=\frac{2\,\lambda^{\alpha+1}}{\Gamma(\alpha+1)}\,x^{2\alpha+1}\;e^{-\lambda x^2}\quad ;x>0,\quad (1)$$

$\lambda>0,\alpha\ge 0$. With $\alpha=0$ we obtain the usual Rayleigh probability density function (pdf).

Initially, this family has been proposed by Voda (1976a, 1976b). We shall name it as the generalized Rayleigh distribution and is denoted by $GRD(\alpha,\ \lambda)$. The two-parameter generalized Rayleigh distribution provides a rich family of specific distributions that have widespread application in many disciplines. Members of this family include the Rayleigh distribution itself, the Half-Normal distribution, the Maxwell distribution, and the Chi-distribution.

The main objective of this paper is to explore the inferential procedures, classical as well as Bayesian, for the generalized Rayleigh distributions.

The Rayleigh distribution itself has had many applications in life testing. Clearly, the GRD family is quite broad and lends itself to widespread application. In the case of reliability modeling, the GRD family is more flexible than the widely used Weibull model, as the latter includes only the Rayleigh distribution as a special case, while the GRD also encompasses the Maxwell and Chi-distributions. In addition, other members of the GRD family, such as the Half-Normal distribution, are quite widely applied in the social sciences and elsewhere.

It is to be noted that most of the cited literature is confined to classical developments and any systematic development on Bayesian results are rarely seen for the generalized Rayleigh distribution. The Bayesian methods are equally well applicable for small sample sizes and censored data problems; the two common features in reliability data analyses.

The advent of Markov chain Monte Carlo(MCMC) sampling has flourished Bayesian statistics. The freely available software package known as Bayesian inference using Gibbs sampling(BUGS) has been in the forefront of this proliferation since the mid-1990s. However, more recent advances in this software, leading first to WinBUGS and now to an open-source version OpenBUGS, Thomas et al. (2006), Thomas (2010) and Lunn et al. (2013), including interfaces to the open-source statistical package R, (R Development Core Team, 2013), have brought MCMC to a wider audience. We shall use OpenBUGS and R software in our present study.

For Bayesian analysis, we also need to assume a prior distribution for the model parameters.. In this paper, Bayesian analysis has been preformed under different loss function assuming independent priors for the parameters.

A major difficulty towards the implementation of Bayesian procedure is that of obtaining the posterior distribution. The process often requires the integration, which is very difficult to calculate not only for high-dimensional complex models even if dealing with low-dimensional models. In such a situation, Markov chain Monte Carlo (MCMC) methods are very useful to simulate the deviates from the posterior density and produce the good approximate results.

The rest of the paper is organized as follows. The generalized Rayleigh distribution and its properties are discussed in Section 2. The point estimation procedures for the parameters of the considered model under classical set-up and the confidence/bootstrap intervals have been constructed in Section 3. In Section 4, we have developed the Bayesian estimation procedure under independent gamma priors for the parameters. To check the applicability of the proposed methodologies, a real data set has been analysed in Section 5. In this section, the ML estimators of the parameters, approximate confidence intervals are presented. We cover Bayesian analysis using the MCMC simulation in Section 6. In this section, the Bayes estimates and credible intervals of parameters, hazard and reliability functions are presented. In Section 7 we have applied the predictive check method in order to give an assessment of the performance of the model for the given data. We have addressed the censored data problem in Section 8. Finally, the conclusions have been given in Section 9.

2. The model

The cumulative distribution function(cdf) of generalized Rayleigh distribution (GRD), corresponding to pdf given in (1), is given by

$$F(x;\alpha,\lambda)=\frac{2\lambda^{2\alpha+1}}{\Gamma(\alpha+1)}\int_0^x x^{2\alpha+1}\,e^{-\lambda x^2}\,dx \tag{2}$$

where $\alpha>0$ and $\lambda>0$ are the parameters.

Define incomplete gamma function as

$$\Gamma(z,a)=\frac{1}{\Gamma(a)}\int_0^z z^{a-1}\,e^{-z}\,dz \tag{3}$$

then we can write

$$F(x;\alpha,\lambda)=\Gamma(\alpha+1,\lambda x^2) \tag{4}$$

This family includes several important probability distributions as special cases.

- For example, if $\alpha=0$ and $\lambda=1/\theta^2$ we obtain the one-parameter Rayleigh distribution with density function
$$f(x;\theta)=\frac{2}{\theta^2}\,x\,\exp\left(-\frac{x^2}{\theta^2}\right)\ ;\ x>0,\ \theta>0.$$
- For $\alpha=1/2$ and $\lambda=1/\theta^2$ we obtain the one-parameter Maxwell distribution with the density function
$$f(x;\theta)=\frac{2}{\theta^3(2\pi)^{1/2}}\,x^2\,\exp\left(-\frac{x^2}{2\theta^2}\right)\ ;\ x>0,\ \theta>0.$$
- For $\alpha=\left((a/2)-1\right)$ and $\lambda=1/2\tau^2$ we obtain the Chi-distribution with 'a' degrees of freedom, whose density function is
$$f(x;\tau,a)=\frac{1}{2^{(a/2)-1}\,\tau^a\,\Gamma(a/2)}\,x^{a-1}\,\exp\left(-\frac{x^2}{2\tau^2}\right);\ x>0,$$
where $a\in N,\ \tau>0$ and N denotes the set of natural numbers.

The reliability/survival function is

$$R(x;\alpha,\lambda)=1-F(x;\alpha,\lambda)\ ;\quad x>0 \tag{4}$$

The hazard rate function is

$$h(x)=\frac{f(x;\alpha,\beta,\lambda)}{1-F(x;\alpha,\beta,\lambda)};\quad x>0. \tag{5}$$

The quantile function is given by

$$F(x;\alpha,\lambda)=p\quad\text{or}\ x_p=F^{-1}(p;\alpha,\lambda) \tag{6}$$

The random deviate can be generated from $GRD(\alpha,\lambda)$ by

$$F(x;\alpha,\lambda)=u\qquad x=F^{-1}(u) \tag{7}$$

where u has the $U(0,1)$ distribution.

The k^{th} moment of $GRD(\alpha,\ \lambda)$ is given by

$$\mu_k^{'} = \Gamma\left(\alpha+\frac{1}{2}k+1\right)\left(\Gamma(\alpha+1)\right)^{-1}\lambda^{-(k/2)}.$$

Therefore,

$$E(X) = \Gamma\left(\alpha+\frac{3}{2}\right)\left(\Gamma(\alpha+1)\right)^{-1}\lambda^{-(1/2)}.\text{ and}$$

$$V(X) = \lambda^{-1}\left[(\alpha+1) - \Gamma^2\left(\alpha+\frac{3}{2}\right)\left(\Gamma^2(\alpha+1)\right)^{-1}\right].$$

The GRD distribution is unimodal and asymmetric. The mode is $((2\alpha+1)/2\lambda)^{0.5}$.

3. Maximum likelihood estimation (MLE)

The nice properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality, make the maximum likelihood estimation most popular and attractive one. In this section, we discuss the maximum likelihood estimators (MLE's) of the $GRD(\alpha,\lambda)$ distribution and discuss their asymptotic properties to obtain approximate confidence intervals based on MLE's.

Let $\underline{x} = (x_1,\ldots,x_n)$ be a sample of size n from $GRD(\alpha,\lambda)$, then the log-likelihood function $\ell(\alpha,\lambda\,|\,\underline{x})$ can be written as;

$$\ell(\alpha,\lambda\,|\,\underline{x}) = n\log 2 + n(\alpha+1)\log\lambda - n\log\Gamma(\alpha+1) + (2\alpha+1)\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}x_i^2 \quad (8)$$

To obtain the MLE's of α and λ, we can maximize (8) directly with respect to α and λ or we can solve the following system of non-linear equations.

$$\frac{\partial\ell}{\partial\alpha} = n\log\lambda - n\,\Psi(\alpha+1) + 2\sum_{i=1}^{n}\log x_i = 0,$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n(\alpha+1)}{\lambda} - \sum_{i=1}^{n}x_i^2 = 0 \quad (9)$$

where $\Psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$ is the digamma function.

Note that the MLEs, respectively $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ cannot be solved analytically. Numerical iteration techniques, such as the Newton-Raphson algorithm, are thus adopted to solve these equations.

3.1 Approximate confidence intervals

The exact distribution of MLEs cannot be obtained explicitly. Therefore, the asymptotic properties of MLEs can be used to construct the confidence intervals for the parameters, Lawless(2003). The asymptotic inference for the parameter vector $\theta = (\alpha,\lambda)$ can be based on the normal approximation of $\hat{\theta} = (\hat{\alpha},\hat{\lambda})$. Under some regular conditions, we have $\sqrt{n}\left(\hat{\theta}-\theta\right) \sim N_2\left(0, I_n(\theta)^{-1}\right)$, for n large, and $I_n(\theta)$ is the per observation expected information matrix. The asymptotic behavior remains valid if $I_n(\theta) = \lim_{n\to\infty} n^{-1}J_n(\theta)$, where $J_n(\theta)$ is the observed information matrix, is replaced by the average sample information matrix evaluated at $\hat{\theta}$, i.e. $n^{-1}J_n(\hat{\theta})$. We have

$$J_n(\theta) = -\begin{pmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\lambda} \\ \frac{\partial^2\ell}{\partial\lambda\partial\alpha} & \frac{\partial^2\ell}{\partial\lambda^2} \end{pmatrix}$$

whose elements are

$$\frac{\partial^2\ell}{\partial\alpha^2} = -n\,\Psi'(\alpha+1) \quad ; \quad \frac{\partial^2\ell}{\partial\lambda^2} = -\frac{n(\alpha+1)}{\lambda^2} \quad \text{and}$$

$$\frac{\partial^2\ell}{\partial\lambda\,\partial\alpha} = \frac{n}{\lambda}.$$

We can use the normal approximation of the maximum likelihood estimator of θ for constructing approximate confidence intervals as well as for testing hypotheses on the model parameters. For example, asymptotic confidence intervals for $\theta = (\alpha,\ \lambda)$ are given, respectively, by $\hat{\alpha} \pm z_{\gamma/2}\,SE(\hat{\alpha})$ and $\hat{\lambda} \pm z_{\gamma/2}\,SE(\hat{\lambda})$, where $SE(\cdot)$ is the square root of the diagonal element of $J_n(\hat{\theta})^{-1}$ corresponding to each parameter, and $z_{\gamma/2}$ is the quantile $100(1-(\gamma/2))\%$ of the standard normal distribution.

3.2 Bootstrap confidence intervals

In this section we propose the confidence intervals based on the bootstrapping. Bootstrap methods are widely used to improve estimators or to build confidence intervals for the parameters. We have used the percentile bootstrap (*Boot-p*) method, proposed by Efron and Tibshirani (1986), to construct confidence intervals for the parameters as well as the reliability and hazard functions. To construct the '*Boot-p*' confidence interval, we proceed as follows, Soliman et al.(2012):

Step 1. From the original data $\underline{x} = (x_1,\ldots,x_n)$ compute the ML estimates $\hat{\alpha}$ and $\hat{\lambda}$ of the parameters: α and λ by solving the nonlinear equations (9).

Step 2. Generate a bootstrap sample $\underline{x}^* = \left(x_1^*,\ldots,x_n^*\right)$ of size n from (1) using $\hat{\alpha}$ and $\hat{\lambda}$. As in Step 1, compute the estimates of α and λ say $\hat{\alpha}^*$ and $\hat{\lambda}^*$, using bootstrap sample.

Step 3. Repeat Step 2, K-times. Obtain the bootstrap estimates $\left(\hat{\alpha}_1^*,\ldots,\hat{\alpha}_K^*\right)$ and $\left(\hat{\lambda}_1^*,\ldots,\hat{\lambda}_K^*\right)$.

Step 4. Let $\left(\hat{\alpha}_{(1)}^*,\ldots,\hat{\alpha}_{(K)}^*\right)$ be the ordered values of the estimates $\left(\hat{\alpha}_1^*,\ldots,\hat{\alpha}_K^*\right)$. The $100(1-\gamma)\%$ two-sided boot-p confidence interval (BCI) for α can be obtained by $\left(\hat{\alpha}_{([K\gamma/2])}^*,\hat{\alpha}_{([K(1-\gamma/2)])}^*\right)$, where $[\chi]$ denotes the largest integer less than or equal to χ. Similarly, we can obtain the $100(1-\gamma)\%$ BCI for λ.

4. Bayesian model formulation

The Bayesian model is constructed by specifying the prior distributions for the model parameters α and λ, and then multiplying with the likelihood function $L(\alpha,\lambda\,|\,\underline{x})$ for the given data $\underline{x}=(x_1,\ldots,x_n)$ to obtain the posterior distribution function using Bayes theorem. The likelihood function is given by

$$L(\alpha,\lambda\,|\,\underline{x})=\frac{2^n\lambda^{n(\alpha+1)}}{(\Gamma(\alpha+1))^n}\left(\prod_{i=1}^n x_i^{2\alpha+1}\right)\exp\left\{-\lambda\sum_{i=1}^n x_i^2\right\}$$

Denote the prior distribution of α and λ as $p(\alpha,\lambda)$. The joint posterior is

$$p(\alpha,\lambda\,|\,\underline{x}) \propto L(\alpha,\lambda\,|\,\underline{x})\; p(\alpha,\lambda)$$

Priors for the parameters

We have assumed independent informative priors for the parameters α and λ. Let us suppose the gamma priors for $\alpha\sim G(a_1,b_1)$ and $\lambda\sim G(a_2,b_2)$ as

$$p(\alpha)=\frac{b^a}{\Gamma(a)}\;\alpha^{a-1}\exp(-b\,\alpha)\quad;\alpha>0,\ a>0,b>0$$

and

$$p(\lambda)=\frac{d^c}{\Gamma(c)}\;\lambda^{c-1}\exp(-d\,\lambda)\quad;\lambda>0,\ c>0,d>0$$

Thus, we have

$$p(\alpha,\lambda)=p(\alpha)\;\;p(\lambda)$$

Posterior distribution

The expression for the posterior, up to proportionality, can be written as

$$p(\alpha,\lambda\,|\,\underline{x}) \propto \frac{\lambda^{n(\alpha+1)+c-1}\;\alpha^{a-1}}{(\Gamma(\alpha+1))^n}\left(\prod_{i=1}^n x_i^{2\alpha+1}\right)\exp\left\{-\left(b\,\alpha+d\,\lambda+\lambda\sum_{i=1}^n x_i^2\right)\right\} \quad (10)$$

The posterior is intractable and no close form inferences are possible. We, therefore, propose to consider MCMC methods to simulate samples from the posterior so that sample-based inferences can be easily drawn. To implement MCMC calculations, Markov chains require a stationary distribution. There are many ways to construct these chains. Several Monte Carlo (MC) based sampling methods for evaluating high dimensional posterior integrals have been developed: MC importance sampling, Metropolis-Hastings sampling, Gibbs sampling, and other hybrid algorithms. A landmark work for Gibbs sampling in problems of Bayesian inference is Gelfand and Smith (1990), which is actually a special case of Metropolis-Hastings sampling, Metropolis et al. (1953) and Hastings (1970).

Gibbs Sampler : Algorithm

It is currently the most popular MCMC sampling algorithm in the Bayesian inference literature. Gibbs sampling belongs to the Markov update mechanism and advocates the philosophy of "divide and conquer." We only need to know the full conditional distributions to apply Gibbs sampling. To carry out Gibbs sampling, the basic scheme is as follows:

Step1: Compute the posterior distribution, upto proportionality, and specify the full conditionals of the model parameters α and λ. The full conditionals of α and λ, using (10), can be written as

- full conditional of α given λ and $\underline{x}$:

$$p(\alpha) \propto \frac{\lambda^{n\alpha}}{(\Gamma(\alpha+1))^n}\left(\prod_{i=1}^n x_i^{2\alpha+1}\right)\;\alpha^{a-1}\,exp(-b\,\alpha)$$

- full conditional of λ given α and $\underline{x}$:

$$p(\lambda) \propto \lambda^{n(\alpha+1)+c-1}\exp\left\{-\lambda\left(d+\sum_{i=1}^n x_i^2\right)\right\}$$

Step 2: Select an initial value $\underline{\theta}^{(0)}=\left(\alpha^{(0)},\lambda^{(0)}\right)$ to start the chain.

Step 3: Suppose at the i^{th}-step, $\underline{\theta}=(\alpha,\lambda)$ takes the value $\underline{\theta}^{(i)}=\left(\alpha^{(i)},\lambda^{(i)}\right)$ then from full conditionals, generate

$\alpha^{(i+1)}$ from $p\left(\alpha\,|\,\lambda^{(i)},\underline{x}\right)$ and

$\lambda^{(i+1)}$ from $p\left(\lambda\,|\,\alpha^{(i+1)},\underline{x}\right)$.

Step 4: This completes a transition from $\underline{\theta}^{(i)}$ to $\underline{\theta}^{(i+1)}$

Step 5: Repeat Step 3, N times.

Posterior sample : MCMC output

Monitor the convergence using convergence diagnostics. Suppose that convergence have been reached after '*B*' iterations (the burn-in period). The MCMC output is referred as the sample after removing the initial iterations (produced during the burn-in period) and considering the appropriate *lag* (or thin interval).

For the posterior analysis, we have the MCMC output (posterior sample) $\left(\underline{\theta}^{(1)},\ldots,\underline{\theta}^{(j)},\ldots,\underline{\theta}^{(M)}\right)$, where

$$\underline{\theta}^{(j)} = \left(\alpha^{(j)},\lambda^{(j)}\right);\ j=1,2,\ldots,M\ .$$

The Bayes estimates of $\underline{\theta}=(\alpha,\lambda)$, under squared error loss function (SELF), are given by

$$\hat{\alpha}=\frac{1}{M}\sum_{j=1}^{M}\alpha^{(j)}\ ;\ \ \hat{\lambda}=\frac{1}{M}\sum_{j=1}^{M}\lambda^{(j)} \qquad (11)$$

We shall use OpenBUGS software to obtain posterior samples. The modular framework of OpenBUGS provides an in depth and interactive analysis of the model with many built-in features and model extensions can easily be accommodated. It is a powerful and flexible tool for Bayesian analysis. BUGS (Bayesian inference Using Gibbs Sampling) is a piece of computer software for the Bayesian analysis of complex statistical models using MCMC methods. OpenBUGS, with open source code, implements MCMC algorithms and is able to analyse highly complex problems for the probability models available in OpenBUGS, Thomas et al.(2006). But model implementation is difficult for the probability distributions, which are not pre-defined in OpenBUGS. Each new model (probability distribution) causes a new software system to be built. Several probability distributions useful in the field of reliability studies are incorporated into *OpenBUGS,* Kumar et al. (2010) and Lunn (2010).

As the generalized Rayleigh distribution is not available in OpenBUGS, it requires incorporation of a module in *ReliaBUGS,* Lunn et al.(2013), which is subsystem of *OpenBUGS*. A module *dgen.rayleighI_T(alpha, lambda)* is written for the generalized Rayleigh, the corresponding computer program can be obtained from authors, to perform full Bayesian analysis in OpenBUGS using the method described in Kumar et al. (2010), Kumar (2010) and Shrestha and Kumar (2013).

5. Data

A real data set is considered for illustration of the proposed methodology. The data extracted from Nichols and Padgett (2006), gives 100 observations on breaking stress of carbon fibres (in Gba)

0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56

5.1. Computation of MLE and Model Validation

The maximum likelihood estimates (MLEs) are obtained by direct maximization of the log-likelihood function $\ell(\alpha,\lambda)$ given in (8) using R software (R Development Core Team, 2013). We consider the Newton-Raphson algorithm in R to compute the MLEs. The contour plot of likelihood is displayed in Figure 1, (+) indicates the ML estimates of α and λ.

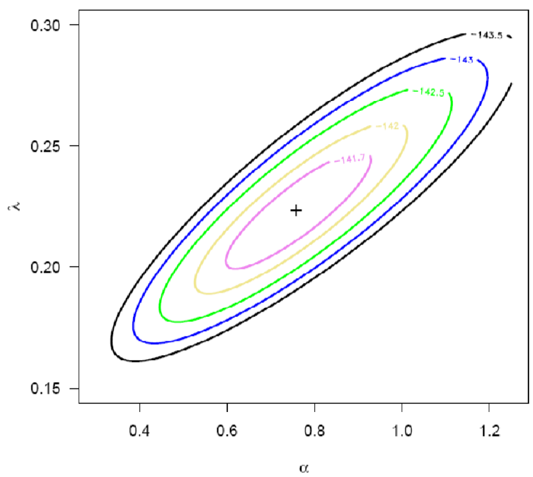


Figure 1: Contour plot of log-likelihood

The value of loglikelihood is $\ell(\hat{\alpha},\hat{\lambda})=-141.437$. The Akaike information criterion (AIC) and Bayesian information criterion(BIC) can be used to determine which model is most appropriate for the given data. For the given data set AIC = 286.874 and BIC = 292.084. The Table 1 shows the ML estimates, standard error(SE) and 95 % Confidence Intervals for parameters α and λ.

Table 1. MLE, standard error and 95% confidence interval (CI)

Parameter	MLE	Std. Error	95% CI
alpha	0.7574	0.22862	(0.3093, 1.2055)
lambda	0.2228	0.03350	(0.1571, 0.2884)

We compute the Kolmogorov-Smirnov (KS) distance between the empirical distribution function and the fitted distribution function when the parameters are obtained by method of maximum likelihood to check the validity of the model. The value of KS statistic is 0.052 and the corresponding p-value is 0.95.

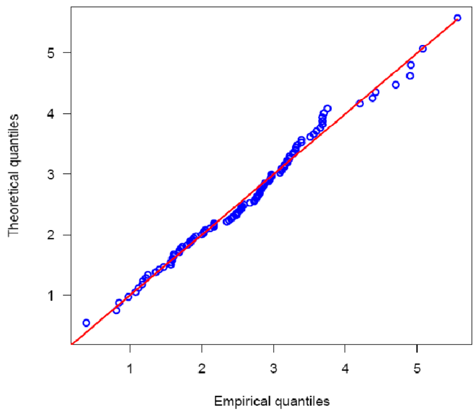


Figure 2: Quantile-Quantile(Q-Q) plot using MLEs as estimate.

The high p-value suggests that fit is satisfactory. The Q–Q plot for the fitted model is shown in Figure 5, Kumar and Ligges(2011). It can be seen that the fitted generalized Rayleigh distribution provides reasonable fit to the given data.

6. Bayesian analysis

We assume the independent gamma priors for $\alpha \sim G(a, b)$ and $\beta \sim G(c, d)$ with hyper-parameter values $(a = b = c = d = 0.001)$. We first construct the contour of un-normailzed joint posterior of (α, λ) in Figure 3, where the contour lines are drawn at 90%, 70%, 40%, 10% and 5% of the maximum value of the posterior density over the grid, Albert(2009). It gives an idea about the parameters.

Script 1 : OpenBUGS *code for the Bayesian analysis*

```
model
{
    for( i in 1 : N )
    {
    x[i] ~ dgen.rayleighI_T(alpha, lambda)
    f[i] <- density(x[i], x[i])
    reliability[i] <- R(x[i], x[i])
    }
# Prior distributions of the model parameters
    alpha ~ dgamma(0.001, 0.001)
    beta ~ dgamma(0.001, 0.001)
}
Data
    list(N=100, c(0.39,...,5.56))
Initial values
    list(alpha= 0.1, lambda= 0.1)   # Chain 1
    list(alpha=2.0, lambda= 2.0)    # Chain 2
```

We run the *Script 1* in OpenBUGS to generate two Markov chains at the length of 40,000 with different starting points of the parameters. We have chosen initial values for the parameters, wide spread over the parameter space, $(\alpha = 0.1,\ \lambda = 0.1)$ for the first chain and $(\alpha = 2.0,\ \lambda = 2.0)$ for the second chain. The convergence is monitored using trace, ergodic mean and BGR plots. It can be observed that the Markov chains reached to the stationary condition very quickly, approximately 2000 iterations. Therefore, burn-in of 5000 samples is more than enough to erase the effect of starting point(initial values). Finally, samples of size 7000 are formed from the posterior by picking up equally spaced every fifth outcome (to minimize the auto correlation among the generated deviates.), i.e. thin=5, starting from 5001.

Therefore, we have the posterior sample from chain 1 and chain 2 as $\left(\alpha_i^{(j)}, \lambda_i^{(j)}\right)$; $j = 1, \ldots, 7000; i = 1, 2.$

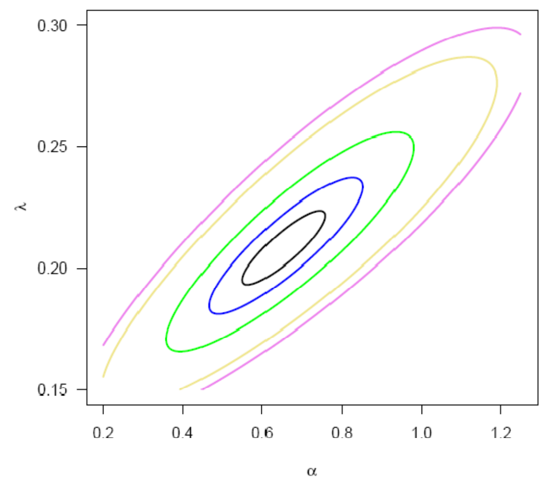


Figure 3: Contour plot of un-normalized joint posterior density of (α, λ).

6.1 Convergence diagnostics

The inferences are valid, if the simulated sample provides a reasonable approximation for the posterior distribution. We have checked the convergence of the simulated draws of (α, λ) for their stationary distributions through different starting points. We have used the graphical diagnostics tools such as: trace, ergodic mean and the Brooks-Gelman-Rubin(BGR) plots. Figure 4 shows the trace, cumulative averages and BGR plots for the parameters alpha and lambda. The trace plots look like a random scatter about some mean value (represented by dotted line). The plots of cumulative averages of sampled values show steady convergence to the mean value (dotted horizontal line). The BGR plots are nice. The ratio of variability between chains to variability within chains is close to one and these are being stable (horizontal) across the width of the plot. We may infer that the chains have escaped from their initial values and found the target distribution of the Markov chain. In fact, these plots are hallmarks of rapid MCMC convergence.

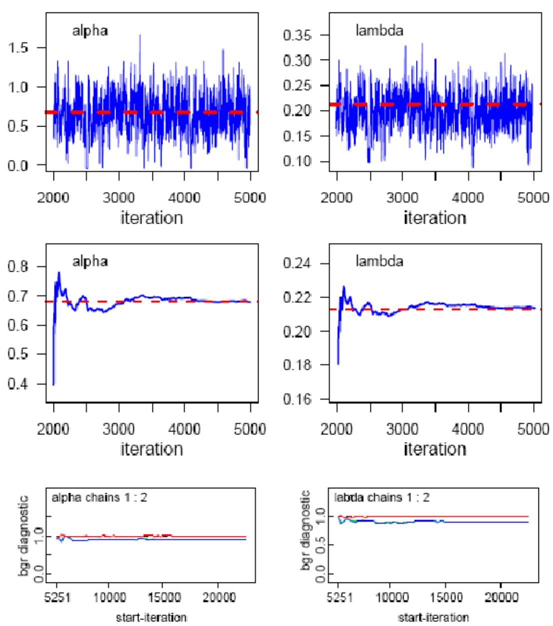


Figure 4: The trace plot (on top), cumulative average plot(in middle) and the BGR plot(at bottom) for alpha and lambda.

From Figure 4, we have no evidence that our posterior samples produced by OpenBUGS chains failed to converge, so we can proceed to use posterior samples for Bayesian inference.

6.2 Posterior analysis

Bayesian analysis is all about the posterior distribution. All of the statistical inferences of a Bayesian analysis come from summary measures of the posterior distribution, such as point and interval estimates. The posterior analysis is presented via quantitative as well as qualitative methods.

6.2.1 Quantitative analysis

The numerical summary is presented for $\left(\alpha_1^{(j)}, \lambda_1^{(j)}\right)$; $j=1,\ldots,7000$ from chain 1. The chain 2 produces the similar results.

We have considered various quantities of interest and their numerical values based on MCMC sample of posterior characteristics for generalized Rayleigh distribution. The MCMC results of the posterior mean, mode, standard deviation(SD), 2.5th percentile, first quartile, median, third quartile, 97.5th percentile, mode and skewness of the parameters α and λ are presented in Table 2. The Bayes estimates under absolute and zero-one loss functions are posterior median and mode, respectively.

Table 2. Numerical summaries based on MCMC sample of posterior characteristics

Characteristics	alpha	lambda
Mean	0.6894	0.2141
Standard Deviation	0.2348	0.0343
2.5th Percentile($P_{2.5}$)	0.2528	0.1503
First Quartile (Q_1)	0.5307	0.1900
Median	0.6826	0.2129
Third Quartile (Q_3)	0.8455	0.2363
97.5th Percentile($P_{97.5}$)	1.1730	0.2840
Mode	0.6685	0.2123
Skewness	0.1302	0.2132

The highest probability density (HPD) credible intervals for α and λ are constructed by using Chen and Shao (1999) algorithm.

Let $(\alpha_{(j)}; j=1,2,\ldots,M)$ be the corresponding ordered MCMC sample of $(\alpha^{(j)}; j=1,2,\ldots,M)$. Then, the $100(1-\gamma)\%$ HPD intervals for α is

$$\left(\alpha_{(k^*)}, \alpha_{(k^*+[(1-\gamma)M])}\right),$$

where k^* is chosen so that

$$\alpha_{\left(k^*+[(1-\gamma)M]\right)} - \alpha_{(k^*)} = \min_{1\le k\le M-[(1-\gamma)M]} \left(\alpha_{(k+[(1-\gamma)M])} - \alpha_{(k)}\right).$$

Here $[\chi]$ denotes the largest integer less than or equal to χ. In the same fashion, one can also obtain the Bayes HPD credible intervals for λ.

Table 3 shows the symmetric credible intervals(SCI) and HPD credible intervals for parameters alpha, beta and lambda. We have also computed the 95% bootstrap confidence interval (BCIs), using the algorithm of the percentile bootstrap method, described in section 3.2, we present the mean of 1000 bootstrap samples of the parameters.

Table 3. Two-sided 95% intervals

Parameter	SCI	HPD	BCI
alpha	(0.253, 1.173)	(0.253, 1.172)	(0.401, 1.377)
lambda	(0.150, 0.284)	(0.145, 0.283)	(0.170, 0.313)

6.2.2 Qualitative analysis

We have considered various graphs for qualitative analysis of the marginal posteriors of the parameters.

These graphs include the boxplot, density strip plot, histogram, marginal posterior density estimate and rug plots for the parameters. We have also superimposed the 95% HPD intervals.

These graphs provide almost complete picture of the posterior uncertainty about the parameters. We have used the posterior sample $\left(\alpha_1^{(j)}, \lambda_1^{(j)}\right);\ j=1,\ldots,7000$ to draw these graphs.

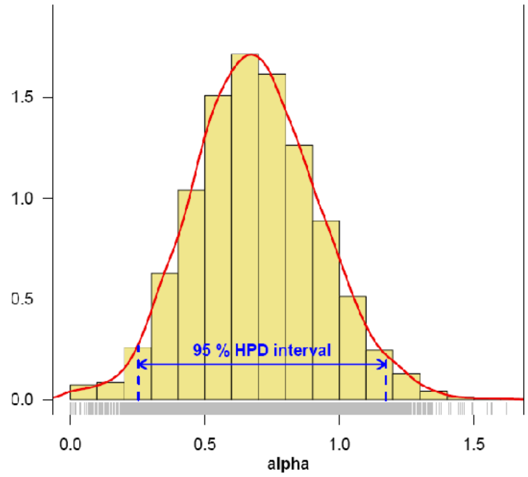


Figure 5(a): Histogram, marginal posterior density and 95% HPD interval

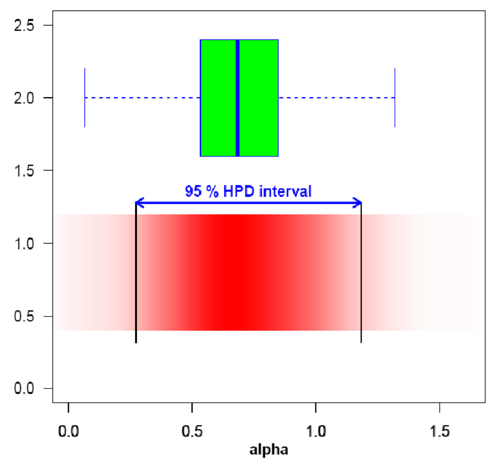


Figure 5(b): Boxplot and density strip plot of α, based on posterior sample.

Jackson (2008) introduced the density strip plot for a univariate distribution as a shaded rectangular strip, whose darkness at a point is proportional to the probability density. It may be noted from Figures 5(b) and 6(b) that density strip plots are more informative as compared to corresponding boxplot.

Probability histogram approximates the marginal posterior distribution. It is the most popular non-parametric method to estimate the density function and gives an idea about skewness, behaviour in the tails, presence of multi-modal behaviour, and data outliers. It may be useful to compare the fundamental shapes associated with standard analytic distributions.

The kernel density estimates have been drawn using R software with the assumption of Gaussian kernel and properly chosen values of the bandwidths. It can be seen that α and λ show positive skewness.

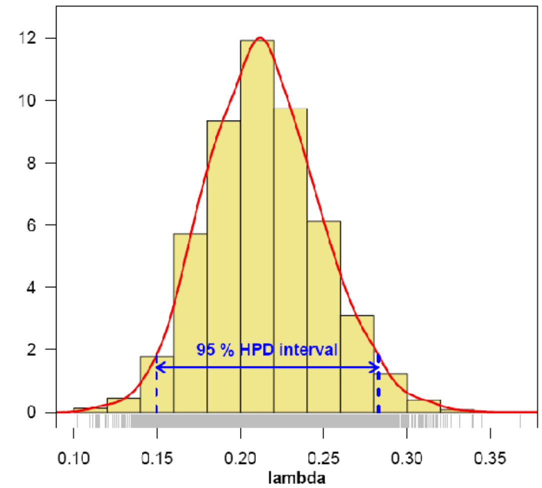


Figure 6(a): Histogram, marginal posterior density and 95% HPD interval based on posterior sample.

Figure 5(a) represents the histogram, marginal posterior density, rug plot and 95% HPD interval for α. The boxplot and the density strip plot are displayed in Figure 5(b).

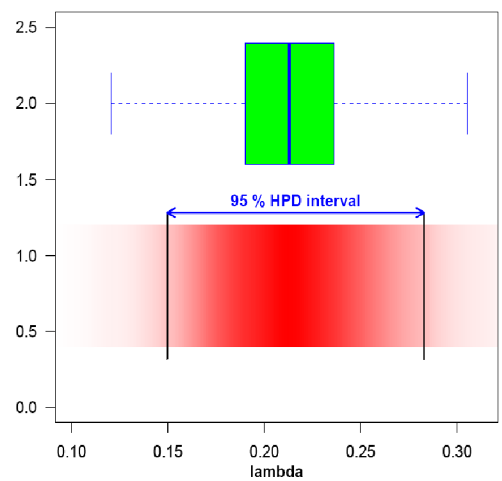


Figure 6(b): Boxplot and density strip plot of λ.

We have plotted the similar graphs in Figure 6(a) and (b) for λ.

6.2.3 Comparison with MLE

For the comparison with ML estimates, we have computed the density function at each observed data point for 7000 posterior samples, $\left(\alpha_1^{(j)}, \lambda_1^{(j)}\right)$; $j = 1, \ldots, 7000$ as

$$f^{(j)}\left(x_i; \alpha_1^{(j)}, \lambda_1^{(j)}\right) \quad ; i = 1, \ldots, 100.$$

In Figure 8 we have plotted 2.5^{th}, 50^{th} and 97.5^{th} quantiles of the estimated density, it can be considered as evaluation of model fit, based on posterior sample.

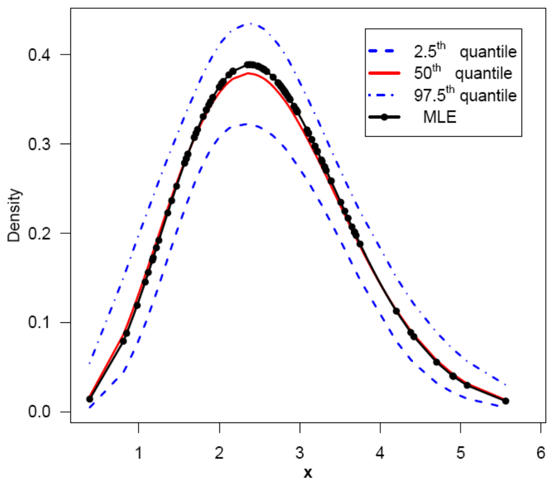


Figure 8: Density estimates

The density corresponding to MLE has been plotted using the ML estimates of the parameters. We observe in the Figure 8, the MLEs and the Bayes estimates are quite close.

6.2.4 Estimation of hazard and reliability functions

The posterior samples may be used to completely summarize the posterior uncertainty about the functions of parameters e.g. reliability and hazard functions. Suppose we wish to give point and interval estimates for reliability and hazard functions at the mission time t=2.41 (at the 40^{th} observed data point).

For the given posterior sample $\left(\alpha_1^{(j)}, \lambda_1^{(j)}\right)$; $j = 1, \ldots, 7000$, we can obtain the posterior sample for the reliability and hazard functions at t=2.41, using (4) and (5), as

$$h^{(j)}\left(x = 2.41; \alpha_1^{(j)}, \lambda_1^{(j)}\right);\ j = 1, \ldots, 7000 \quad \text{and}$$

$$R^{(j)}\left(x = 2.41; \alpha_1^{(j)}, \lambda_1^{(j)}\right);\ j = 1, \ldots, 7000.$$

The MCMC results of the posterior mean, mode, standard deviation (SD), first quartile, median, third quartile, mode, skewness, 95% symmetric credible intervals(SCI) and HPD credible intervals of reliability and hazard functions are displayed in Table 4.

Table 4. Posterior summary for Reliability and Hazard functions at t=2.41

Characteristics	Reliability	Hazard
Mean	0.5451	0.6979
Standard Deviation	0.0392	0.0736
First Quartile (Q_1)	0.5188	0.6470
Median	0.5453	0.6946
Third Quartile (Q_3)	0.5720	0.7447
Mode	0.5441	0.6859
Skewness	-0.0460	0.2583
95% SCI	(0.4682, 0.6226)	(0.5623, 0.8516)
95% HPD	(0.4662, 0.6199)	(0.5572, 0.8459)

The ML estimates of reliability and hazard function at t=2.41 are computed using invariance property of the MLE. The ML estimates are $\hat{h}(t = 2.41) = 0.7061$ and $\hat{R}(t = 2.41) = 0.5505$.

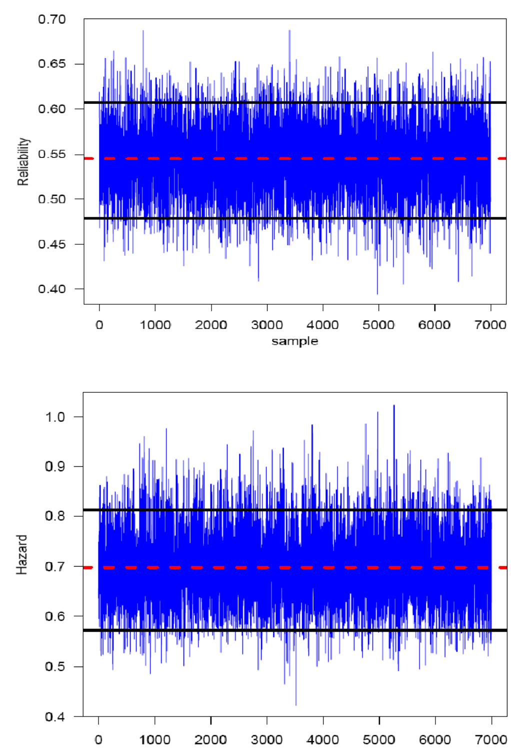


Figure 10: MCMC output of $R(t = 2.41)$ and $h(t = 2.41)$. Dashed line(...) represents the posterior median and solid lines(-) represent lower and upper bounds of 95% probability intervals (HPD)

A trace plot is a plot of the iteration number against the value of the draw of the parameter at each iteration. Figure 10 displays 7000 chain values for the hazard $h(t = 2.41)$ and reliability $R(t = 2.41)$ functions, with their sample median and 95% HPD credible intervals.

The 95% percentile bootstrap confidence interval (BCIs) for reliability and hazard function at t=2.41, using the algorithm described in section 3.2, based on 1000 bootstrap samples are (0.5378, 0.5693) and (0.6329, 0.7933), respectively.

Now we shall demonstrate the effectiveness of proposed methodology for the entire data set. Since we have an effective MCMC technique, we can estimate any function of the parameters. For this, we have computed the reliability function given in (4) at each data point, using posterior sample $\left(\alpha_1^{(j)}, \lambda_1^{(j)}\right)$; $j = 1,\ldots,7000$

$$R^{(j)}\left(x_i\,; \alpha_1^{(j)},\, \lambda_1^{(j)}\right); i = 1,\ldots,100; j = 1,\ldots,7000.$$

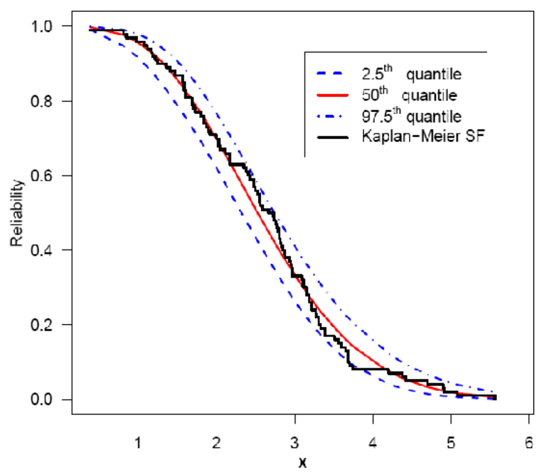


Figure 11: Reliability function estimate using MCMC and Kaplan-Meier estimate

The Figure 11, exhibits the estimated reliability function (dashed line: 2.5^{th} and 97.5^{th} quantiles; solid line : 50^{th} quantile) using Bayes estimate based on MCMC output. We have superimposed the Kaplan-Meier estimate of the reliability function to make the comparison more meaningful. The Figure 11 shows that reliability estimate based on MCMC is very close to the empirical reliability estimates.

7. Posterior predictive analysis

A Bayesian approach for checking whether the model fits the data is known as posterior predictive checking. To do posterior predictive checking, we generate replicates of the dataset from the predictive distribution and compare these replicate datasets to the sample. If the replicate datasets and the sample are similar, we conclude that the model fits the data, (Gelman 2003) and (Gelman et al. 2004). Modern Bayesian computational tools, however, provide straightforward solutions as one can easily simulate predictive samples if MCMC outputs are available from the posterior corresponding to the assumed model. Let $\underline{x} = (x_1,\ldots,x_n)$ is a vector of n observations from the model $GRD(\alpha,\lambda)$. We can simulate the posterior predictive distribution as

- Obtain posterior sample $\underline{\theta}^{(j)} = \left(\alpha^{(j)}, \lambda^{(j)}\right)$; $j = 1,2,\ldots,M$
- For each posterior sample $\theta^{(j)}$, simulate n data points, as $x_i^{rep,j} \sim GRD\left(\alpha^{(j)}, \lambda^{(j)}\right)$; $j = 1,2,\ldots,M$ and $i = 1,2,\ldots,n$

Thus, for each sampled value, $\left(\alpha^{(j)}, \lambda^{(j)}\right)$, we obtain M replicated data set $\underline{x}^{rep,j} = \left(x_1^{rep,j},\ldots,x_n^{rep,j}\right)$.

The predictive analysis is based on 2000 posterior samples. For this purpose, 2000 samples have been drawn from the posterior using MCMC procedure and then obtained predictive samples from the model under consideration using each simulated posterior sample. In fact, we have 2000 replicates for each data point $x_i\,; i = 1,\ldots,100$. The graphical method is one of the best way to assess model adequacy based on posterior predictive distributions. We view the model-checking as a comparison of the data with the replicated data given by the model, which includes exploratory graphics, Chaudhary and Kumar(2013).

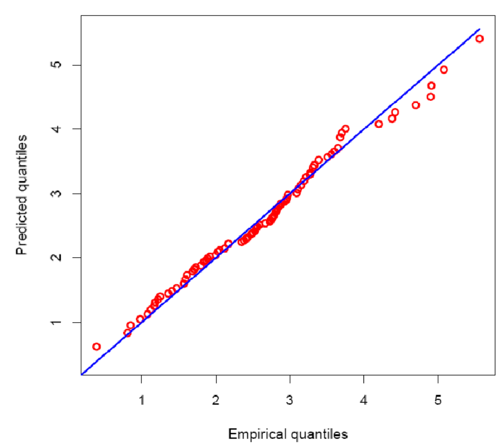


Figure 12: Q-Q plot of predictive quantiles versus empirical quantiles

Figure 12 represents the Q-Q plot of predicted quantiles *vs.* observed quantiles. The estimate of CDF based on replicated data given by the model is displayed in Figure 13. Figure 13 exhibits graphical posterior predictive check of the model adequacy, solid line(—) represents the posterior median and dashed lines(...) represent lower and upper bounds of 95% probability intervals, empirical distribution function is superimposed.

We, therefore, conclude that the generalized Rayleigh distribution is compatible with the given data set.

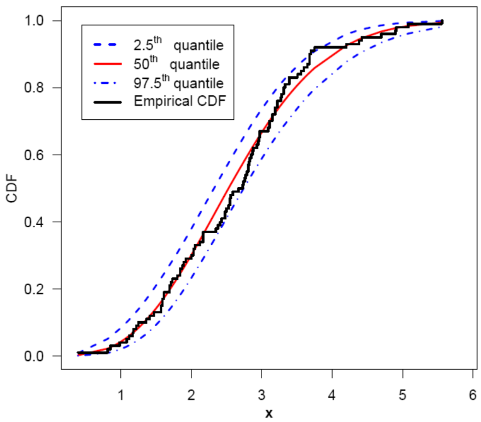


Figure 13: Estimate of CDF based on predicted values

To obtain further clarity on our conclusion for the study of model compatibility, we have considered plotting of density estimates of second largest and largest

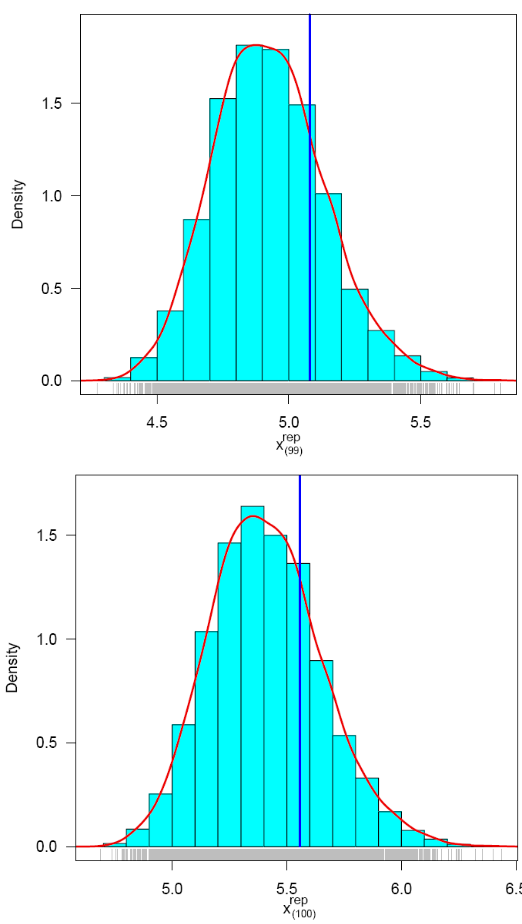


Figure 14: Posterior predictive densities of $x_{(99)}^{rep}$ and $x_{(100)}^{rep}$, vertical lines represent corresponding observed value

i.e. $\left(x_{(99)}^{rep} \text{ and } x_{(100)}^{rep}\right)$ replicated future observations from the model with superimposed corresponding observed data, Figure 14.

As the Figure 14 shows, the posterior predictive distributions are centered over the observed values, which indicate good fit. In general, the distribution of replicated data appears to match that of the observed data fairly well.

The Table 5 shows the MCMC results of the posterior mean, median, mode and 95% HPD credible intervals for

$$\left(x_{(2)}^{rep}, x_{(30)}^{rep}, x_{(99)}^{rep} \text{ and } x_{(100)}^{rep}\right)$$

Table 5. Posterior characteristics

	Observed	Mean	Median	Mode	HPD
$x_{(2)}^{rep}$	0.81	0.83	0.90	0.85	(0.640, 1.014)
$x_{(30)}^{rep}$	2.00	2.04	2.11	2.06	(1.857, 2.223)
$x_{(99)}^{rep}$	5.08	4.93	5.06	4.88	(4.517, 5.340)
$x_{(100)}^{rep}$	5.56	5.41	5.56	5.35	(4.943, 5.890)

Overall, the results of the posterior predictive simulation indicate that model fits these data particularly well. Model fit assessments based on posterior predictive checks should not be used for model selection, Ntzoufras (2009).

8. Bayesian analysis for Type-II censored data

In this section, we shall address the problem of Bayesian analysis of type-II censored data from $GRD(\alpha, \lambda)$. Let us consider that the last four observations of the data set, given in section 5, are censored so that only the first 96 observations are available for analysis. The problem we wish to consider is that of estimating the parameters α and λ, as well as predicting the future four failure times $(x_{97:100}, x_{98:100}, x_{99:100}, x_{100:100})$. Assume the independent gamma priors for the parameters. The straightforward solution may be obtained by making slight modifications in Script 1.

The OpenBUGS code used to analyze this problem is shown in Script 2. In Script 2, it may be noted that in *model* section the likelihood is written for uncensored and censored observations separately, and in *data* section censored observations are treated as NA(Not Available).

We used two distinct sets of initial values to start the Markov chain for the model parameters. We let OpenBUGS to generate the initial values for the censored observations. We have monitored α and λ as well as future four failure times $(x_{97:100}, x_{98:100}, x_{99:100}, x_{100:100})$, where the $x_{97:100}$ is written as x.pred[1] in Script 2. The 30000 iterations are generated from each chain. We discarded the first 5000(burn-in) MCMC iterations and used the remaining 25000. In order to reduce the autocorrelation within the MCMC series for the correlation parameter we used every 5th MCMC iteration for posterior computations.

Script 2 : OpenBUGS *code for the censored case*

```
model
{
   for( i in 1 : N - 4 )  # uncensored observations
   {
   x[i] ~ dgen.rayleighI_T(alpha, lambda)
   }
    for(i in N - 3 : N)  # censored observations
   {
    x[i] ~ dgen.rayleighI_T(alpha, lambda) C(x[N - 4], )
    }
  for(i in 1 : 4) # predicted failure times of censored items
    {
     x.pred[i] <- ranked(x[N- 3 : N], i)
   }
   # Prior distributions of the model parameters
        alpha ~ dgamma(0.001, 0.001)
        beta ~ dgamma(0.001, 0.001)
   }
   Data
        list(N=100, c(0.39,...,4.70, NA,NA,NA,NA))
   Initial values
        list(alpha= 0.1, lambda= 0.1)   # Chain 1
        list(alpha=2.0, lambda= 2.0)    # Chain 2
```

Therefore, we have the posterior sample from chain 1 and chain 2 as

$$\left(\alpha_i^{(j)}, \lambda_i^{(j)}, x_{97:100}^{(j)}, x_{98:100}^{(j)}, x_{99:100}^{(j)}, x_{100:100}^{(j)}\right);$$

$j = 1,\ldots,5000;\ i = 1,2.$ We have also checked the convergence of the sequences of α and λ for their stationary distributions through different starting values. It was observed that the Markov chains reached to the stationary condition very quickly.

We have considered chain 1 for posterior inference. Figure 15 shows the trace plot of α and λ for 5000 sampled prevalence values after burn-in. The plot shows good mixing of the Gibbs sampler. The posterior mean is represented by dashed line where as solid lines represent 95% HPD confidence intervals for the parameters. The quantitative measures based on posterior sample are summarized in Table 6. The histogram, marginal posterior density and 95% HPD for α and λ are plotted in Figure 16 and Figure 17, respectively. Similar graphs for censored failures are depicted in Figure 18.

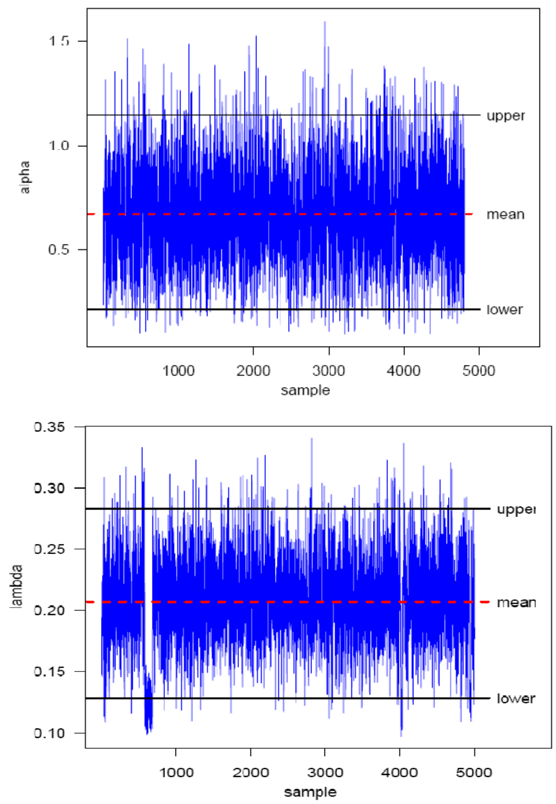


Figure 15: Trace plot of α and λ. Dashed line(...) represents the posterior mean and solid lines(-) represent lower and upper bounds of 95% probability intervals (HPD)

Table 6. Numerical summaries based on MCMC sample of posterior characteristics

Characteristics	alpha	lambda
Mean	0.6718	0.2072
Standard Deviation	0.2370	0.0379
First Quartile (Q_1)	0.5079	0.1830
Median	0.6627	0.2064
Third Quartile (Q_3)	0.8256	0.2319
Mode	0.6625	0.1965
Skewness	0.2736	0.0125
95% SCI	(0.227, 1.164)	(0.128, 0.284)
95% HPD	(0.215, 1.146)	(0.128, 0.283)

Table 7. Posterior summary based on MCMC sample

	Mean	Median	Mode	95% HPD
$x_{97:100}$	4.84	4.90	4.74	(4.70, 5.11)
$x_{98:100}$	5.03	5.14	4.86	(4.71, 5.49)
$x_{99:100}$	5.29	5.47	5.10	(4.76, 5.98)
$x_{100:100}$	5.76	6.06	5.41	(4.85, 6.87)

The Table 7 shows the MCMC results of the posterior mean, median, mode and 95% prediction intervals for censored failure times $\left(x_{97:100}, x_{98:100}, x_{99:100}, x_{100:100}\right)$.

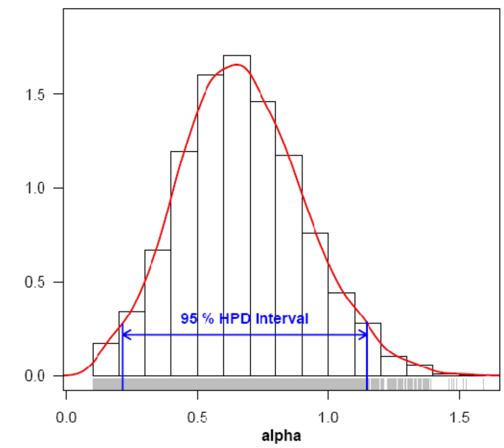


Figure 16: Histogram, marginal posterior density and 95% HPD interval based on posterior sample.

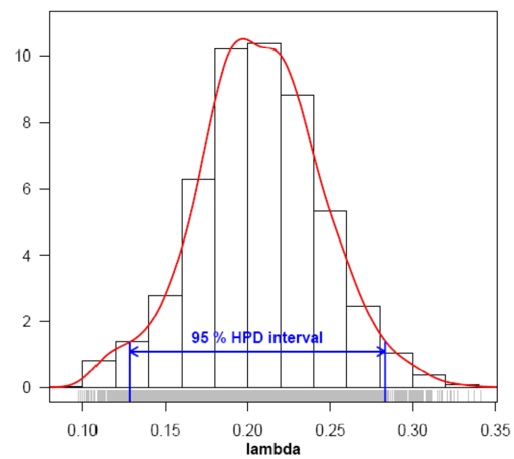
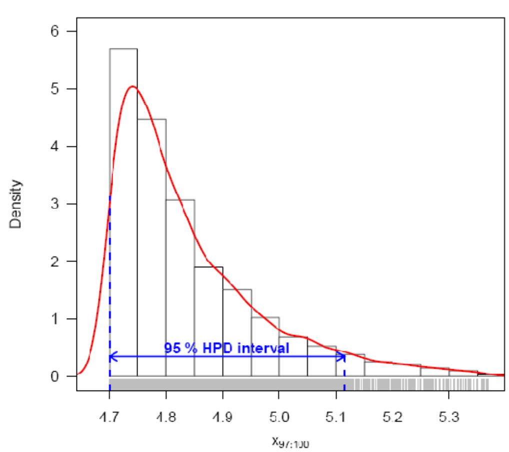


Figure 17: Histogram, marginal posterior density and 95% HPD interval based on posterior sample.



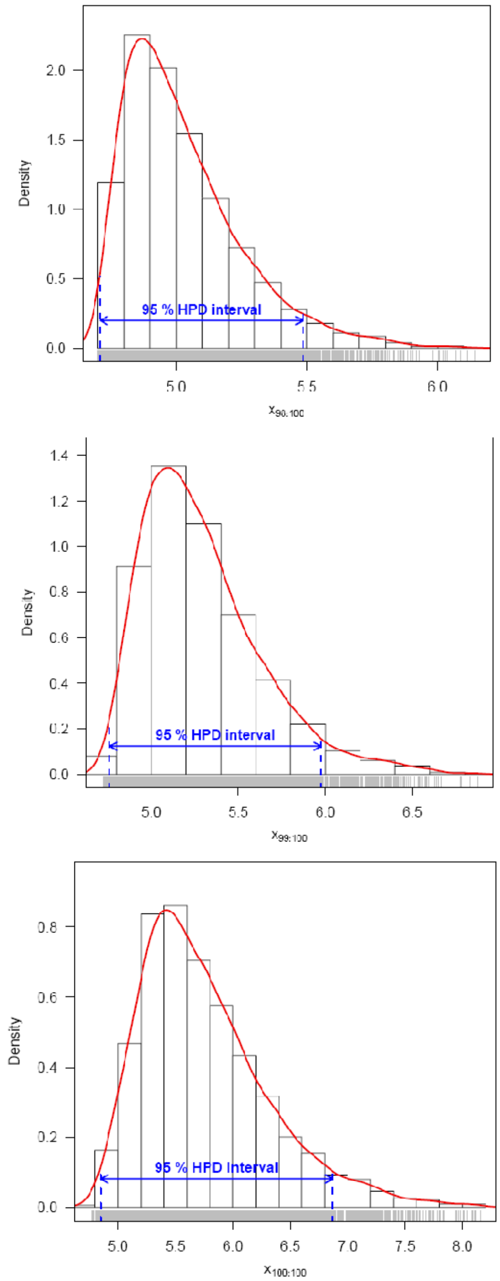


Figure 18: Histogram, posterior predictive densities and 95% HPD intervals of $\left(x_{97:100}, x_{98:100}, x_{99:100}, x_{100:100}\right)$ based on posterior sample

We have observed that the Gibbs sampling technique can be used quite effectively, for estimating the posterior predictive density and also for constructing predictive interval in case of censored sample.

9. Conclusion

The methods described to implement modern computational- based classical as well as Bayesian approaches related to generalized Rayleigh distribution.

We have proposed an integrated procedure for Bayesian inference using MCMC methods. We obtain the Bayes estimates and the corresponding credible intervals using Gibbs sampling procedure for complete as well as for type-II censored data. We have obtained the estimates and probability intervals for parameters, hazard and reliability functions. We have presented the model compatibility analysis via the posterior predictive check method. We have applied the developed techniques on a real data set.

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