Fixed Point Theorem for Mapping Satisfying a Contractive Condition of Integral Type in D-Metric Spaces

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Research Article

Abstract: In this paper a fixed point theorem for mappings satisfying a contractive inequality of integral type in generalized metric space is established. This result is analogous to the result of Branciari.

Introduction: Dhage [1] introduced the notion of D-metric space (Generalized Metric Space) as follows.

A non-empty set X together with a function $D : X \times X \times X \to R^+$, R^+ denote the set of all non-negative real numbers, called a D-metric on X, becomes a D-metric space (X, D) if D satisfies the following properties :

(i) D(x, y, z) = 0 if and only if x = y = z (coincidence)

(ii) D(x, y, z) = D(x, z, y) = ….. (symmetry)

(iii) $D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality)

Example: Define a function $D :: X \times X \times X \to R$ by D(x, y, z) = max {d(x, y), d(y, z), d(z, x)} for all $x, y, z \in X$ and where d is an ordinary metric on X. Then D defines a D-metric on X.

A sequence $\{x_n\}$ in a D-metric space (X, D) is said to be D-convergent and converges to a point $x \in X$ if $\lim_{m,n} D(x_m, x_n, x) = 0$

A sequence $\{x_n\}$ in (X, D) is said to be D-Cauchy if $\lim_{m,n,p} D(x_m, x_n, x_p) = 0$. A complete D-metric space X is one in which every D-Cauchy sequence converges to a point in it.

In a recent paper Branciari [2] established the following theorem.

Theorem : Let (X, d) be a complete metric space, $C \in [0, 1)$, $f : X \to X$ a mapping such that, for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(t)dt \le C \int_0^{d(x,y)} \phi(t)dt$$

where $\phi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative, and such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)\,dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_n f^n x = z$.

The aim of this paper is to prove the above result of Branciari [2] in Generalized Metric Space.

Theorem: Let X be a complete D-metric space, $k \in [0, 1)$, $T : X \to X$ a mapping such that, for each $x, y, z \in X$,

$$\int_0^{D(Tx,Ty,Tz)} \phi(t)dt \le k \int_0^{D(x,y,z)} \phi(t)dt \quad \ldots(1)$$

where $\phi : R^+ \to R^+$ is a Lebesgue – integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$,

$$\int_0^{\varepsilon} \phi(t)dt > 0, \quad \ldots(2)$$

Then T has a unique fixed point $u \in X$ such that, for each $x \in X$, $\lim_n T^n x = u$.

Proof :- Let $x \in X$, and for brevity, define $x_n = T^n x$

For each integer $m \ge n \ge 1$, from (1)

$$\int_0^{D(x_n,x_{n+1},x_m)} \phi(t)dt \le k \int_0^{D(x_{n-1},x_n,x_{m-1})} \phi(t)dt$$

$$\le \ldots\ldots \le k^n \int_0^{D(x_0,x_1,x_{m-n})} \phi(t)dt \quad \ldots(3)$$

Taking the limit of (3), as $n \to \infty$, yields

$$\lim_{n} \int_0^{D(x_n, x_{n+1}, x_m)} \phi(t)dt = 0 \quad \ldots(4)$$

which, from (2), implies that

$$\lim_{n} D(x_n x_{n+1}, x_m) = 0 \quad \ldots(5)$$

We now show that $\{x_n\}$ is Cauchy. Suppose that it is not. Then there exists an $\varepsilon > 0$ and subsequences $\{m(k)\}$, $\{n(k)\}$, $\{p(k)\}$ such that

$k \leq m(k) < p(k) < n(k)$

$D(x_{m(k)}, x_{n(k)}, x_{p(k)}) \geq \varepsilon$, $D(x_{m(k)}, x_{n(k)-1}, x_{p(k)-1}) < \varepsilon \quad \ldots(6)$

Using the triangular inequality and (6),

$D(x_{m(k)-1}, x_{n(k)-1}, x_{p(k)-1}) \leq D(x_{m(k)-1}, x_{m(k)}, x_{p(k)-1}) + D(x_{m(k)-1}, x_{n(k)-1}, x_{m(k)}) + D(x_{m(k)}, x_{n(k)-1}, x_{p(k)-1})$
$< D(x_{m(k)-1}, x_{m(k)}, x_{p(k)-1}) + D(x_{m(k)-1}, x_{n(k)-1}, x_{m(k)}) + \varepsilon \quad \ldots(7)$

Using (5) and (7)

$$\lim_{k} \int_0^{D(x_{m(k)-1}, x_{n(k)-1}, x_{p(k)-1})} \phi(t)dt \leq \int_0^{\varepsilon} \phi(t)dt \quad \ldots(8)$$

Using (1), (6) and (8), it then follows that

$$\int_0^{\varepsilon} \phi(t)dt \leq \int_0^{D(x_{m(k)}, x_{n(k)}, x_{p(k)})} \phi(t)dt$$

$$\leq k \int_0^{D(x_{m(k)-1}, x_{n(k)-1}, x_{p(k)-1})} \phi(t)dt \leq k \int_0^{\varepsilon} \phi(t)dt$$

which is a contradiction. Therefore $\{x_n\}$ is Cauchy, hence convergent. Call the limit u.

From (1)

$$\int_0^{D(Tu, x_n, x_{n+1})} \phi(t)dt \leq k \int_0^{D(u, x_{n-1}, x_n)} \phi(t)dt \quad \ldots(9)$$

Taking the limit of (9) as $n \to \infty$, we obtain

$$\int_0^{D(Tu,u,u)} \phi(t)dt \leq k \int_0^{D(u,u,u)} \phi(t)dt$$

which implies that

$$\int_0^{D(Tu,u,u)} \phi(t)dt = 0$$

which from (2), implies that D(Tu, u, u) = 0 or Tu = u. This implies u is a fixed point of T.

For uniqueness,

Suppose that u and v are fixed point of T. Then from (1),

$$\int_0^{D(u,u,v)} \phi(t)dt = \int_0^{D(Tu,Tu,Tv)} \phi(t)dt \leq k \int_0^{D(u,u,v)} \phi(t)dt$$

which implies that

$$\int_0^{D(u,u,v)} \phi(t)dt = 0$$

which, from (2), implies that D(u, u, v) = 0 or u = v, and the fixed point is unique.

Example: Let X = [0, 2] and D be a D-metric on X defined by D(x, y, z) = max{d(x, y), d(y, z), d(z, x)}, where d is a usual metric on X. Define T : $X \to X$ such that $T(x) = \frac{x+1}{2}$. Let $\phi : R^+ \to R^+$ be such that $\phi(t) = t$, then all the conditions of theorem are satisfied and Pr $\in [1/4, 1)$ and clearly 1 is the unique fixed point of T.

References:

[1] Dhage, B.C., Generalized metric spaces and mappings with fixed point, Bull. Cal. Math. Soc. **84**, 329, 1992.

[2] Branciari, A., A fixed point theorem for mappings satisfying a general contractive condition of Integral type, Int. J. Math. Math. Sci. **29**, no. 9, 531, 2002.