Research Article

Sasakian Metric as Yamabe Soliton

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Abstract In this paper we study Sasakian manifold whose metric is as Yamabe soliton and obtain some result.
Key words: Sasakian metric, Yamabe soliton, cconstant scalar cuvature, infinitesimal automorphism.
Mathematical Subject Classification 2010: 53C25, 53C21, 53C44.

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Received Date: 28/08/2014 Accepted Date: 09/09/2014

Access this article online	
Quick Response Code:	Website: www.statperson.com
	DOI: 17 September 2014

INTRODUCTION

For a smooth Riemannian manifold (M, g_o), the evolution of the metric g_o in the t to $g = g(t)$ through the equation:
$\frac{\partial}{\partial t} g = -rg, g(0) = g_0$
(r denotes the scalar curvature of g), is known as Yamabe flow, and was introduced by Hamilton [4]. The significance of Yamabe flow lies in the fact that is a natural geometry deformation to metric of constant scalar curvature. One notes that Yamabe flow corresponds to the fact diffusion case of the porous medium equation (the plasma equation) in Mathematical physics. Just as Ricci soliton is a special solution of the Ricci flow, A Yamabe soliton is a special solution of Yamabe flow that moves by one parameter family of diffeomorphism ϕ_t generated by a fixed (time independent) vector field V on M, and homotheties, i.e. $g(.,t) = \sigma(t)\phi_t^* g_0$. Equivalently, a Yamabe soliton is defined on a Riemannian manifold (M, g) (suppressing the subscript 0 in g) by a vector field V satisfying;

$$£_V g = (c - r)g, \tag{1}$$

where $£_V$ denotes the derivative operator along V, r is the scalar curvature (not necessarily constant) of g, and the constant $c = \sigma(0)$ (see Chow et al. [3]). Recently, Sharma and Ghosh [6] studied a non-trivial Ricci soliton as a Sasakian manifold M and show M is a homothetic to the standard Sasakian metric on the Heisenberg nill[3]. The purpose of this paper is to study a Yamabe soliton as a Sasakian metric and prove the following results,

Theorem: If a Sasakian metric on a manifold M is a Yamab soliton, then it has constant scalar curvature, and the flow vector field is killing. Further V is also an infinitesimal automorphism of the contact metric structure on M.
As indicated in [3], the scalar curvature of a Yamabe soliton on a compact manifold is constant. Theorem 1 replaces the compactness with Sasakian condition, and provides more in the conclusion. At this point, we would like to point out that

How to site this article: Ankita Rai, Dhruwa Narain. Sasakian Metric as Yamabe Soliton. *International Journal of Statistika and Mathemtika* Aug-Oct 2014; 11(2): 97-99. http://www.statperson.com (accessed 17 September 2014).

eq. (1) define a conformal vector field V with conformal scale function $c-r$, and that a Sasakian manifold with a non-isometric conformal vector field. We also point out that we have assumed the initial metric g to be Sasakian, however the subsequent metrics $g(t)$ along the Yamabe flow need not be Sasakian.

Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be contact manifold if it carries a global 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M. Given a contact 1-form η there exists a unique vector field ξ such that $(d\eta)(\xi,X)=0$ and $\eta(\xi)=1$. Polarizing $d\eta$ on the contact subbundle $\eta=0$, one obtain a Riemannian metric g and a (1,1)-tensor field ϕ such that,

$$(d\eta)(X,Y)=g(X,\phi Y), \phi\xi=0, \eta(X)=g(X,\xi), \phi^2=-I+\eta\otimes\xi, \tag{2}$$

g is called a contact metric associated with η. A vector field V on a contact metric manifold is said to be an infinitesimal contact transformation (see Tanno [7]) if $£_V\eta=f\eta$ for some smooth function f, and is said to be an infinitesimal automorphism of a contact metric structure if it is leaves all the structure tensors η,ξ,ϕ and g invarriant (see Tanno [8]).

A contact metric is said to be K-contact if ξ is Killing with respect to g. A contact metric manifold (M,g) is Sasakian if the cone manifold $(C(M),\bar{g})=(M\times R^+, t^2g+dt^2)$ is Kahler, Sasakian metric are K-contact and K-contact 3-dimensional metrics are Sasakian. For Sasakian manifold,

$$\nabla_X\xi=-\phi X \tag{3}$$

$$(\nabla_X\phi)Y=g(X,Y)\xi-\eta(Y)X, (div\phi)X=-2n\eta(X), \tag{4}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, Q\xi=2n\xi, \tag{5}$$

where ∇, R and Q denotes respectively, the Riemannian connection and curvature tensor and the (1,1)-tensor metrically equivalent to the Ricci tensor of g.

Let us now briefly review conformal vector fields. A vector field on an m-dimensional Riemannian manifold (M,g) is said to be conformal if,

$$£_Vg=2\rho g, \tag{6}$$

for a smooth function ρ on M. A conformal vector satisfies,

$$(£_VS)(X,Y)=-(m-2)g(\nabla_XD\rho,Y)+(\Delta\rho)g(X,Y), \tag{7}$$

$$£_Vr=-2\rho r+2(m-1)\Delta\rho, \tag{8}$$

where D is the gradiant operator and $\Delta=-divD$ is the Laplacian operator of g. For detail we refer to [9]. Further there is a lemma (we refer to Sharma [10]) which is, for a Sasakian manifold, (a). $(£_V\eta)(\xi)=\frac{c-r}{2}$ and (b). $\eta(£_V\xi)=\frac{r-c}{2}$.

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The Ricci tensor S of a Sasakian metric is given by,

$$S(X,Y)=\alpha g(X,Y)+\beta\eta(X)\eta(Y), \tag{9}$$

where α and β are constant. As V is a conformal vector field with $\rho=\frac{c-r}{2}$, equ. (7) and (8) can be written as,

$$(£_VS)(X,Y)=\frac{m-2}{2}[g(\nabla_XDr,Y)-(\Delta r)g(X,Y)], \tag{10}$$

$$£_Vr=-(m-1)\Delta r-r(c-r), \tag{11}$$

taking Lie-derivative of (9) along V and using equ. (1), (10) and (11) we obtains,

$$\left(\frac{m-2}{2}\right)g(\nabla_XDr,Y)=\left[\left(\frac{m-2}{2}\right)(\Delta r)+\alpha(c-r)\right]g(X,Y)+\beta[(£_V\eta)(X)\eta(Y)+(£_V\eta)(Y)\eta(X)], \tag{12}$$

As ξ is killing, we have $\xi r=0$. Differentiating it along an arbitrary vector field X and using (2) gives,

$$g(\nabla_XDr,\xi)=(\phi X)r,$$

Substituting ξ in place of Y in (12) and using the above equation and lemma provides the equation,

$$\left(\frac{m-2}{2}\right)(\phi X)r=\left[\left(\frac{m-2}{2}\right)(\Delta r)+\left(\alpha+\frac{\beta}{2}\right)(c-r)\right]\eta(X)+\beta(£_V\eta)(X), \tag{13}$$

Substituting ξ for X in the above equation, using (2) and lemma, we immediately get,

$$\left(\frac{m-2}{2}\right)\Delta r=(\alpha+\beta)(r-c), \tag{14}$$

In view of the equation (14) and (13) shows that,

$$\beta(£_V\eta)(X)=\left(\frac{m-2}{2}\right)(\phi X)r-\beta\left(\frac{r-c}{2}\right)\eta(X), \tag{15}$$

From (14) and (15), equation (13) becomes,

$$\left(\frac{m-2}{2}\right)\nabla_X Dr = \beta(r-c)[X-\eta(X)\xi] - \left(\frac{m-2}{2}\right)g(X,\phi Dr)\xi - \eta(X)\phi Dr, \quad (16)$$

At this point we assume that $(e_i), (i = 1,2,\dots\dots..2n+1)$ to be local orthonormal frame on M. Using (3.8) we compute,

$$S(X,Dr) = g(R(e_i,X)Dr,e_i),$$

And then using (2.1), (2.2), skew-symmetry of ϕ and the second equation of (2.3) we obtain,

$$S(X,Dr) = -\eta(X)g\left(\phi\nabla_{e_i}Dr,e_i\right),$$

Where i is the summer over $i = 1,2,\dots\dots..(2n+1)$. The use of (16) in the right hand side of the foregoing equation show that $S(X,Dr) = 0$, using this in (9) immediately yield $\alpha Xr = 0$, which gives $Xr = 0$. Hence we conclude that r is constant. From (14) we find that $r = c$. Thus from (1) we find that $£_V g = 0$, i.e V is killing. From equation (13) we conclude that $£_V\eta = 0$. As V is killing, we also conclude that $£_V\xi = 0$. Finally, taking Lie-derivative of first equation in (2) along V and noting that Lie-derivative commutes with exterior derivative, we conclude that $£_V\phi = 0$. Thus, V is an infinitesimal automorphism of the contact metric structure of M.

Theorem: If a Sasakian metric on a manifold M is a Yamabe soliton, then it has constant scalar curvature, and the flow vector field is killing. Further V is also an infinitesimal automorphism of the contact metric structure on M.

Corollary: For 3-dimensional Sasakian metric on a manifold M is a Yamabe soliton, then it follow the above theorem, also it is Einstein manifold and constant scalar curvature of constant curvature 1.

Proof: For three dimensional Sasakian manifold Ricci tensor is defined by,

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\},$$

Apply above theorem we get, $r = constant$ and when $r = 6$ then by above equation we get,
$S(X,Y) = 2g(X,Y)$ i.e $S = 2g$. Hence $M$is Einstein. Also by (Sharma and Blair [5]), M is scalar curvature of constant curvature 1. This complete the proof.

Remark

It is evident from the conclusion $£_V\xi = 0$ of theorem that, if g is not of constant curvature, and V is point-wise non-collinear with ξ, then the pair (V,ξ) span a foliation, and ϕV is normal to those leaves. From equ. (3) we have $\nabla_V\xi = -\phi V$. Using this and denoting the Riemannian connection induced on a leaf $\sum$ by D, we find $D_V\xi = 0$. Also, as $\nabla_\xi\xi = 0$, Gauss equation implies that $D_\xi\xi = 0$ and ξ is an asympatic direction. A straightforward computation show that the sectional curvature of $\sum$ with respect to plane section spanned by V and ξ vanishes. Hence $\sum$ is intrinsically flat. Furthermore, the conclusion $£_V\xi = 0$ implies the existence of a function f on M such that $V = f\xi - \frac{1}{2}\phi Df$ (see [1]). Since V is killing, we find that $\xi f = 0$. If V is point wise collinear with ξ, then it follow that V is a constant multiple of ξ. On the other hand, V cannot be orthogonal to ξ unless $V = 0$, we also observe that $Vf = 0$, consequently, Df is orthogonal to ξ and V, and hence normal to $\sum$.

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Source of Support: None Declared
Conflict of Interest: None Declared