Research Article

Determination of various fractal dimensions in Logistic Map

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Abstract In this paper we consider the one dimensional chaotic Logistic map $f(x) = \mu x(1 - x)$, $\mu\epsilon(0\,,4\,], x\epsilon[\,0,1\,]$ where μ is the control parameter and study various fractal dimensions like Box counting dimension (D_b), Information dimension (D_I) and Correlation dimension (D_c) . We see that $D_c \le D_I \le D_b$ as required.
Key words: Logistic map, fractal, box counting dimension, information dimension, correlation dimension.

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Received Date: 28/09/2014 Accepted Date: 05/10/2014

Access this article online	
Quick Response Code:	Website: www.statperson.com
	DOI: 06 October 2014

INTRODUCTION

1. Logistic Map is defined by $f(x) = \mu x(1 - x)$, where μ is the control parameter. For decades, several iterated functions have been extensively studied, and rich contents have been explored .Logistic map is one of the well known maps and has became a standard map for studying iteration. This map contains all the interesting subjects in non-linear dynamics.[1-9] In general, the values of x and μ of logistic map are restricted in the range, $0 \le x \le 1\,, 0 < \mu \le 4$ so that each x in the interval [0,1] is mapped onto the same interval [0,1]. It is known that there is a stable fixed point x*=0 in the range $0 \le \mu \le 1$, and another stable fixed point x*=1-1/μ in the range $1 \le \mu \le 3$.After that ,we have period-doubling bifurcation at μ=3, 3.4494897, 3.54409 .... .These numerical results are well known and are easy to reproduce on computer. However, it is a puzzle why we have two neighbour regions, 0≤x≤1 and 1≤μ≤3, that each has a stable fixed point of f. According to sharkovsky ordering[1], the appearance of the order of periods should be $1 \to 2 \to 4 \to 8 \to \cdots \cdots$In logistic map the accumulation point is given by $\mu = 3.569945672 \ldots ....$

2. The word "fractal" was coined by Mandelbrot in his fundamental essay from Latin fractus, meaning broken, to describe objects that were too irregular to fit into a traditional geometrical setting.[14] Chaotic dynamical systems exhibit trajectories in their phase space that converges to a strange attractor. A quantitative measure of strangeness of the attractor is called fractal dimension.[21] We are familiar with the idea that a point is zero dimensional, a straight line is one dimensional, a square is two dimensional, the dimension of a cube is three. What is the dimension of a strange attractor? And why we need it to calculate? The dimension of the attractor is the first level of knowledge to characterise its properties. Generally speaking, we may think of the dimension as giving in some way, the amount of information

How to site this article: H K Sarmah, B K Bhattacharyya, Anil Kr Jain. Determination of various fractal dimensions in Logistic Map. *International Journal of Statistika and Mathemtika* Aug-Oct 2014; 11(3): 202-206. http://www.statperson.com (accessed 06 October 2014)

necessary to specify the position of the point on the attractor to within a given accuracy.[15] In order to determine the geometric behaviour of the attractor set at accumulation point , we calculate various fractal dimensions viz box counting dimension, information dimension, correlation dimension.[13]

3. BOX COUNTING DIMENSION[14]: One of the most widely used, due to its easy mathematical calculation and empirical estimation since 1930.Although it has got various names like Kolmogrov entropy, entropy dimension, capacity dimension etc., we call it as box counting dimension and define as[14] Let F be any non empty bounded subset of R^n and let $N\delta(F)$ be the smallest number of sets of diameter at most δ which can cover F. The lower and upper box counting dimensions of F respectively are given by

$\underline{\dim B F} = \lim_{\delta\to 0} \frac{log N\delta(F)}{-log\delta}$ and

$\overline{\dim B F} = \lim_{\delta\to 0} \frac{log N\delta(F)}{-log\delta}$

If these are equal , we refer to the common value as the box counting dimension of F.

$\dim_B F = \lim_{\delta\to 0} \frac{log N\delta(F)}{-log\delta}$.

Where $\delta>0$ is taken as sufficiently small to ensure that $-\log\delta$ and similar other quantities are strictly positive. We consider box dimension only for non empty bounded sets to avoid problems with log0 and log∞.

4. INFORMATION DIMENSION:[13,20,21] Many researcher have suggested alternative definition of fractal dimension which is similar to box dimension where researcher tries to account for the frequency with the trajectory visits each covering cube. To calculate the information dimension, one counts the number of points N_i in each of the cube N and determines the probability of finding a point in that cube P_i where $P_i = N_i / N_0$, $\sum_{i=0}^{n} \text{Pi} = 1$,where N_0 is the total number of points in the set. In our case the attractor set is obtained after iterating at some initial point at the parameter value $\mu = 3.569945672$ and $N_0 \neq N$. The information entropy (Shanon entropy) is defined by the expression $I(\epsilon) = -\sum Pi \log_2 pi$ where $I(\epsilon)$ has unit of bits for small ϵ we define

$D_I = \frac{I(\epsilon)}{\log(1/\epsilon)} = \frac{\sum Pi \log_2 pi}{\log_2 \epsilon}$

Where I is termed as Information dimension.

Note: this definition is related to box counting dimension in the following way.

If the probability of all the cubes are equal then $\sum pi = N\,P_i = 1$ and so $P_i = 1/N$

$I(\epsilon) = -\sum Pi \log_2 pi = -N.\frac{1}{N}\log_2\frac{1}{N} = log_2 N$.

And hence $D_I = D_b$, where D_b is the box counting dimension.

In general it can be shown that $D_I \le D_b$ .Thus information dimension gives a lower bound of box counting dimension.

5.CORRELATION DIMENSION:[16,18,20] As box counting dimension converges very slowly i.e. it takes time in partition the state space with cubes and then to locate the trajectory points within the cubes, so we need another algorithm for calculating dimension of the attractor . Therefore we provide a most computationally efficient and relatively fast algorithm for dimension estimation (in Grassberger and Procaccia, 1983, way) which is popularly known as correlation dimension. To define correlation dimension (Dc) we first need to define correlation sum C_R as follows

$$C_R = \frac{1}{N}\sum_{J=1}^{N}\left[\frac{1}{N-1}\sum_{K=1,K\neq J}^{N}\varphi(R-|x_j-x_k|)\right]$$

Where φ is a Heavyside step function given as

$\varphi\,(|(R - |xj - xk|) = 1\ if |\,xj - xk| < R$

$= 0$ if $|\,\mathrm{xj} - \mathrm{xk}| > R$ The correlation sum C_R behaves as a power of R for small R $C_R \propto R$^D_c Hence the correlation dimension D_c is given as $D_c = \lim\ R \to 0\ \frac{logCR}{logR}$. Further it can be shown that $D_c \le D_I \le D_b$.Thus D_c gives a lower bound for information as well as box counting dimension[13,16,20,21]

6. GENERALISED DIMENSION: In computing the box-counting dimension, one either counts or does not count a box according to whether there are some points or no points in the box. No provision is made for weighting the box count according to how many points are inside a box. In other words, the geometrical structure of the fractal set is analyzed but the underlying measure is ignored [22].

The order-q Renyi entropy is defined as $Iq(R) = \frac{1}{q-1}\ log \sum_{i=1}^{N(R)} pi$^q ----(A) and hence the generalized dimension (i.e. Renyi dimension) is defined as Dq=lim$R$→0$\frac{1}{q-1}\frac{log \sum_{i=1} p_i^q}{logR}$ -------------------(B) Where $p\ i$ is the probability which is defined as pi=$\frac{Ni}{N}$, N= number of iterated points in the ith cube of side R , N=total no. of iterated points. It is also known as generalised box counting dimension. Now we discuss what happens when q=0 and q=1. Let us consider q=0.in this case the sum log$\sum_{i=1}$ pi^q =1 gives the logarithm of the number of nonempty cubes used to cover the attractor set. In other words, the Renyi dimension D_0 is nothing but our box counting dimension. Let us now consider q≈1.For this value of q ,we cannot use the equation (B) directly as denominator 1-q vanishes. So we take the help of limit property i.e. taking q→1, ie .we use L'Hospital rule $\lim_{q\to 1}$ Iq (R) $= \frac{\frac{d}{dq}(log \sum_{i=1} pi^q)}{\frac{d}{dq}(q-1)} = \frac{\sum pi^q logpi}{\sum p_i^q}$ =$\sum$ pi logpi , Since $\sum pi$ =1.

Hence D_1 =lim R→0$\frac{\sum pilogpi}{logR}$, this is known as information dimension. Next we will show that D_2 is just the correlation dimension. To see this connection, let us look at the probability sum. (Where the summation is taken over the number of boxes N(R)) $S_q\ (R) = \sum P_i^q$ (Where the summation is taken over the number of cubes N(R))

$=\sum P_i P_i^{q-1}$

$=> S_q(R) = \frac{1}{N}\sum_{j=1}^{N}\left[\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}\varphi\left(R-\left|x_j-x_k\right|\right)\right]^{q-1}$

Where we replace $p_i = \frac{1}{N}$ and$P_i^{q-1} = \left[\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}\varphi\left(R-\left|x_j-x_k\right|\right)\right]^{q-1}$

Thus the generalised correlation sum for N points is given as follow

$$\mathrm{Gq\ (N,R)} = \left[\frac{1}{N}\sum_{j=1}^{N}\left[\frac{1}{N-1}\sum_{k=1,k\neq j}^{N}\varphi\left(R-\left|x_j-x_k\right|\right)\right]^{q-1}\right]^{\frac{1}{q-1}}$$

Where $\varphi\left(R-\left|x_j-x_k\right|\right) = 1$ if $\left|x_j-x_k\right| < R$

$= 0$ if $\left|x_j-x_k\right|$>R

Also the generalised dimensionD_q, in terms of generalised correlation sum

$D_q = \lim_{R\to 0}\frac{1}{q-1}\frac{\log \mathrm{Cq\ (N,R)}}{\log R}$ ----------------- (D)

Moreover $\lim_{N\to\infty} \mathrm{Gq\ (N,R)} = \mathrm{Gq\ (R\,)}$

From relation (D) we can see that Dq can be calculated by calculating the generalized correlation sum for any value of q. In particular by calculating the generalized correlation sum for q= 0, 1, 2, we can get box counting, information and correlation dimensions respectively. In the next part we take the help of relation (D) to calculate the three dimensions.

OUR ESTIMATED RESULT

6.1.Box Counting dimension: here we consider q=0,so that (D) gives us box counting dimension .we have calculated this dimension at the parameter value $\mu = 3.569945672$ and iterated the difference map N=30000 times .The parts of the plotted points (log R, log Gq (R)) which follows equation (D) is taken. The points in the scaling region are as follows –

{-5.75646,-4.34843},{-5.5262,-4.23557},{-5.29595,-4.08264},{-5.06569,-3.96113}, {-4.83543,-3.85013},{-4.60517,-3.73719},{-4.37491,-3.5729},{-4.14465,-3.47004}, {-3.9144,-3.36381},{-3.68414,-3.23772}, {-3.45388,-3.08544},{-3.22362,-2.98181} {-2.99336,-2.86402},{-2.7631,-2.74663},{-2.53284,-2.57565},{-2.30259,-2.4796}, {-2.07233,-2.36914},{-1.84207,-2.22931}, {-1.61181,-2.09357},{-1.38155,-2.00799}}.

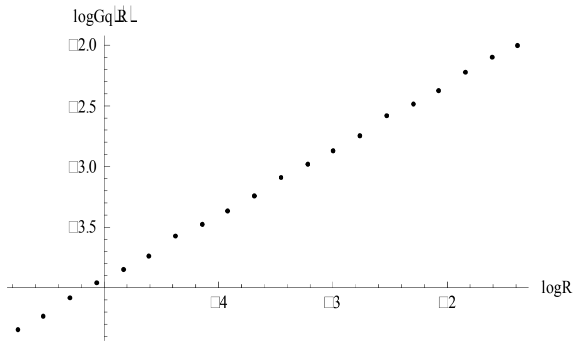

Figure 1: logR-log G_q(R) graph obtained for q=0 .Abscissa represents logR and ordinent represents log G_q(R).

The slope of the graph when fitted a straight line in the scaling region is 0.538852 with the mean deviation 0.0832474,which may be called as the box counting dimension .

6.2. Information dimension: here we take q very near to 1 ,At q=1.0000001,so that (D) gives us Information dimension. We have calculated this dimension at the parameter value $\mu = 3.569945672$ and iterated the map N=30000 times .The parts of the plotted points (log R, log Cq (R)) which follows equation (D) is taken. The points in the scaling region are as follows

{-7.82879,-5.22946},{-7.59853,-5.14845},{-7.36827,-4.99606},{-7.13801,-4.8613}, {-6.90776,-4.76286},{-6.6775,-4.66306},{-6.44724,-4.53185},{-6.21698,-4.38692}, {-5.98672,-4.28361},{-5.75646,-4.1759},{-5.5262,-4.05598},{-5.29595,-3.91856}, {-5.06569,-3.80785},{-4.83543,-3.69762},{-4.60517,-3.58991},{-4.37491,-3.43327}, {-4.14465,-3.32835},{-3.9144,-3.22288},{-3.68414,-3.09967},{-3.45388,-2.96496}, {-3.22362,-2.86637},{-2.99336,-2.75051},{-2.7631,-2.64228},{-2.53284,-2.47835}, {-2.30259,-2.3726},{-2.07233,-2.26304},{-1.84207,-2.12884},{-1.61181,-2.00607}, {-1.38155,-1.92897},{-1.15129,-1.81377},{-0.921034,-1.70218},{-0.690776,-1.53885}, {-0.460517,-1.41129},{-0.230259,-1.29357}.

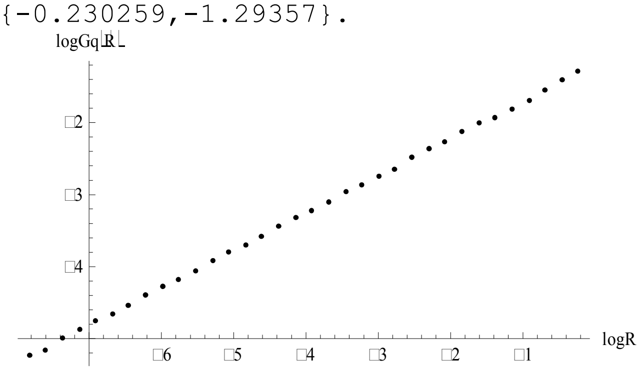

Figure 2: logR-logG_q(R) graph obtained for q=1.0000001 .Abscissa represents logR and ordinent represents logG_q(R).

The slope of the graph when fitted a straight line in the scaling region is 0.517147 with the mean deviation 0.0728649, which may be called as the information dimension.

6.3. Correlation dimension: Here we consider q=2,so that (D) gives us correlation dimension. we have calculated this dimension at the parameter value $\mu = 3.569945672$ and iterated the map N=30000 times .The parts of the plotted points (log R, log Gq (R)) which follows equation (D) is taken. The points in the scaling region are as follows –

{-6.90776,-4.57959},{-6.6775,-4.47866},{-6.44724,-4.36521},{-6.21698,-4.22042}, {-5.98672,-4.1166},{-5.75646,-4.01154},{-5.5262,-3.89191},{-5.29595,-3.76849}, {-5.06569,-3.66882},{-4.83543,-3.55926},{-4.60517,-3.45686},{-4.37491,-3.30592}, {-4.14465,-3.19756},{-3.9144,-3.08905},{-3.68414,-2.96577},{-3.45388,-2.84651}, {-3.22362,-2.761},{-2.99336,-2.64996},{-2.7631,-2.54968},{-2.53284,-2.39224}, {-2.30259,-2.274},{-2.07233,-2.1615},{-1.84207,-2.02921},{-1.61181,-1.91768}.

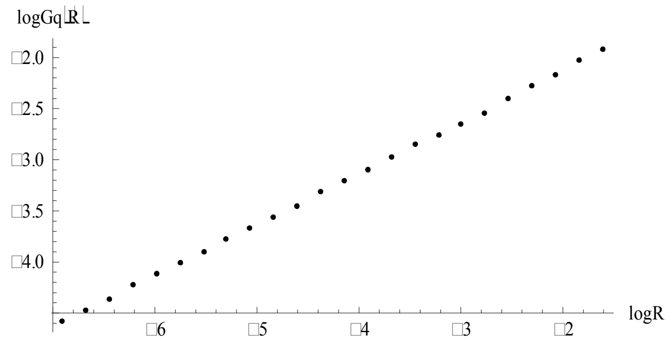

Figure 3: logR-logG_q(R) graph obtained for q=2. Abscissa represents logR and ordinent represents logG_q(R).

The slope of the graph when fitted a straight line in the scaling region is 0.500677 with the mean deviation 0.0553417., which may be called as the correlation dimension. Thus we see that, Correlation dimension Dc =0.500677, Information dimension D_I = 0.517147, Box counting dimension D_b = 0.538852, Clearly $D_c \leq D_I \leq D_b$.

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Source of Support: None Declared
Conflict of Interest: None Declared