Research Article

Standard projective simplicial kernels and the second abelian cohomology of topological groups

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Abstract Let A be an abelian topological G-module. We give an interpretation for the second cohomology, $H^2(G,A)$, of G with coefficients in A. As a result we show that if P is a projective topological group, then $H^2(P,A)=0$ for every abelian topological P-module A.
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INTRODUCTION

Let G be a topological group and A an abelian topological group. It is said that A is an abelian topological G-module, whenever G acts continuously on A. For all $g \in G$ and $a \in A$ we denote the action of g on a by ${}^g a$. Suppose that A is an abelian topological G-module. Hu[4] showed that if G is Hausdorff, then there is a one to one correspondence between the second cohomology, $H^2(G,A)$, of G with coefficients in A and the set, $Ext_s(G,A)$, of all equivalence classes of topological extensions with continuous sections. Thus, $H^2(G,A)$, induces a group structure on $Ext_s(G,A)$ and consequently, under this group product, $H^2(G,A)$, is isomorphic to $Ext_s(G,A)$. It is known that $Ext_s(G,A)$ with the Baer sum is an abelian group and the Baer sum on $Ext_s(G,A)$ coincides with the group product induced by $H^2(G,A)$. In other words, if G is Hausdorff then, $H^2(G,A)$ is isomorphic to $Ext_s(G,A)$ with the Baer sum[1]. In section 2, we prove that every Markov (Graev) free topological group is a projective topological group and also we show that projectivity in the category of topological groups is equivalent to $\mathcal{F}$-projectivity in the sense of Hall[3]. In addition, we

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introduce the notion of a standard simplcial kernel of a topological group G. Also, we define the group $Opext_s(G,A)$, and we conclude that, there is a canonical isomorphism, between $H^2(G,A)$ and $Opext_s(G,A)$ (without Hausdorffness of G). In section 3, we give a characterization of $H^2(G,A)$, when A is an abelian topological G-module. As a result, we show that if P is a projective topological group then $H^2(P,A)=0$, for every abelain topological P-module A.

PROJECTIVE SIMPLICIAL KERNELS

In this section, we introduce the notion of a standard projective simplicial kernel of a topological group G.

Definition 2.1.: *A topological group P is said to be projective if, for every continuous epimorphism $\pi : A \to B$, where π has a continuous section, and for every continuous homomorphism $f : P \to B$, there exists a continuous homomorphism $g : P \to A$ such that the following diagram is commutative.*

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{g} & \downarrow f & & \\ A & \xrightarrow{\pi} & B & \to & 1 \end{array}$$

If X is a topological space, then the (Markov) free topological group over X is the pair $(F_X, \sigma_X : X \to F_X)$ in which F_X is a group equipped with the finest group topology such that σ_X is continuous. Such a topology always exists[8], and has the following universal property: every continuous map f from X to an arbitrary topological group G lifts to a unique continuous homomorphism $\bar{f}$, .i.e., $\bar{f}\sigma_X = f$. Similarly, one can define the Graev free topological group, (F_X^*, σ_X^*), over a pointed topological space (X,e). For information on free (abelian) topological groups see[2,6,8]. The following facts about Markov (Graev) free toplogiacal group are well-known:

1. *The Markov (Graev) free topological group over a (pointed) topological space X ((X,e)) is the free group with the same canonical map over the (pointed) set X ((X,e));*
2. *F_X (F_X^*) is Hausdorff if and only if X is functionally Hausdorff;*
3. *σ_X (σ_X^*) is a homeomorphic embedding if and only if X is completely regular;*
4. *σ_X (σ_X^*) is a closed homeomorphic embedding if and only if X is Tychonov.*

The elements of free group, F_X, over a set X is denoted by $|x_1|^{\epsilon_1} \dots |x_n|^{\epsilon_n}$, where $\epsilon_i = \pm 1$. This notation is useful whenever X is a group.

Lemma 2.2. *Every Markov (Graev) free topological group is a projective topological group. Proof.* Assume that F is a Markov free topological group over the space X, and let $\pi : A \to B$ be a continuous epimorphism having a continuous section s and $f : F \to B$ a continuous homomorphism. Then, there is a unique continuous homomorphism $\bar{f} : F \to A$ such that the following diagram is commutative.

$$\begin{array}{ccccc} X & \xrightarrow{\sigma_X} & F & \xrightarrow{f} & B \\ \downarrow \sigma_X & & & & \downarrow s \\ F & & \overset{\bar{f}}{\dashrightarrow} & & A \end{array}$$

Hence, $sf = \bar{f}$ on X. So, $\pi\bar{f} = \pi(sf) = (\pi s)f = f$ on X. Since X generates the group F, then it is easy to see that $\pi\bar{f} = f$. Consequently, F is a projective topological group. Now, Let X be a topological space with a fixed point $e \in X$. Assume that F^* is the free topological group in the sense of Graev over (X,e), and let $\pi : A \to B$ be a continuous epimorphism with continuous section $s : B \to A$. We can assume that s is a normal section, i.e., $s(1) = 1$, since it is enough to define the new continuous section $\tilde{s} : B \to A$ by $\tilde{s}(b) = s(b)s(1)^{-1}$. In fact, since e is the neutral element of F^*, then $f(e)=1$. Thus, $sf(e)=1$. Hence there exists a unique continuous homomorphism $\tilde{f}:F^*\to A$ such that $sf=\tilde{f}$ on X. The rest of the proof is the same as the first part. □ Note that the free functor $F_{(-)}$ from the category, T, of topological spaces to the category, TG, of topological groups is left adjoint to the forgetful functor, $G:\mathrm{TG}\to\mathrm{T}$. Let $\mathcal{F}$ be the class of all continuous epimorphisms having a continuous section. Thus, by [3, Theorem 2] one can see the following:

Remark 2.3. *A topological group P is $\mathcal{F}$-projective if and only if P is projective.*

Definition 2.4. *Let M, F and G be topological groups, and let $\iota_0,\iota_1:M\to F$ and $\tau:F\to G$ be continuous homomorphisms. It is said that (M,ι_0,ι_1) is a simplicial kernel of τ, whenever $\tau\iota_0=\tau\iota_1$ and it has the following universal property:*
if $j_0,j_1:K\to F$ are continuous homomorphisms and $\pi j_0=\pi j_1$, then there exists a unique continuous homomorphism $h:K\to M$ such that $j_0=\iota_0 h$ and $j_1=\iota_1 h$, i.e., h commutes the following diagram.

$$\begin{array}{ccc} K & \overset{j_0}{\underset{j_1}{\rightrightarrows}} & F \\ {\scriptstyle h}\downarrow & & \| \\ M & \overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}} & F \end{array}$$

Definition 2.5. *Let (M,ι_0,ι_1) be a simplicial kernel of $\tau:P\to G$. The quadruplet (M,ι_0,ι_1,τ) is said to be a projective simplicial kernel of G, if P is a projective topological group and τ is a continuous (open) epimorphism which has a continuous section $s:G\to P$. If P is a Markov (Graev) free topological group, then (M,ι_0,ι_1,τ) is called a Markov (Graev) simplicial kernel of G.*

Lemma 2.6. *If a continuous homomorphism $f:G\to H$ has a continuous section, then f is an open epimorphism.*

Proof. The proof is a standard argument. □

For any topological group G there exists at least one projective simplicial kernel of G. Since the identity map $Id_G:G\to G$ lifts to a unique homomorphism $\tau_G:F_G\to G$. Thus, $\tau_G\sigma_G=Id_G$. The uniqueness property implies that τ_G is a map by the following rule:

$\tau_G:F_G\to G$, $|g_1|^{\epsilon_1}\dots|g_n|^{\epsilon_n}\mapsto g_1^{\epsilon_1}\dots g_n^{\epsilon_n}$ where $\epsilon_i=\pm1$.

The map τ_G is called *multiplication map*. Obviously, $\sigma_G:G\to F_G$, $g\mapsto|g|$, is a continuous section for τ_G. By Lemma 2.6, τ_G is an open continuous epimorphism. We take $M_G=\{(x,y)\,|\,x,y\in F_G,\tau_G(x)=\tau_G(y)\}\subset F_G\times F_G$. Hence, M_G has subspace topology induced by the product topology $F_G\times F_G$. Suppose $\iota_0=\iota_0^G,\iota_1=\iota_1^G:M_G\to F_G$ are the canonical projection maps, where $\iota_0^G(x,y)=x$ and $\iota_1^G(x,y)=y$. It is easy to see that $(M_G,\iota_0^G,\iota_1^G)$ is a simplicial kernel of τ_G. Hence, $(M_G,\iota_0^G,\iota_1^G,\tau_G)$ is a projective simplicial kernel of G and we call it the *standard Markov simplicial kernel of* G. By a similar way we can define the *standard Graev (projective) simplicial kernel of* G.

Suppose that (M,ι_0,ι_1,τ) is a projective simplicial kernel of G. We sometimes denoted it by the following

$M\overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}}P\xrightarrow{\tau}G\to1.$

If $(M_G,\iota_0,\iota_1,\tau_G)$ is a standard projective simplicial kernel of G, then put $\Delta_P=\{(x,x)\,|\,x\in P\}\subset M$. Now, let A be an abelian topological G-module. Obviously, A is a topological P-module via τ and a topological M-module via $\tau\iota_0$ (or $\tau\iota_1$). Define $Z^1(M,A)=\{\alpha\,|\,\alpha\in Der_c(M,A),\alpha(\Delta_P)=0\}$. It is clear that $Z^1(M,A)$ is a subgroup of abelian group $Der_c(M,A)$. Consider $\eta:Der_c(P,A)\to Z^1(M,A)$, $\alpha\mapsto\alpha\iota_0\alpha\iota_1^{-1}$. Obviously, η is well-defined and since A is an abelian group, then

$$\eta(\alpha.\beta)=(\alpha.\beta\iota_0)(\alpha.\beta\iota_1^{-1})=(\alpha\iota_0.\beta\iota_0)(\alpha\iota_1^{-1}.\beta\iota_1^{-1})$$
$$=(\alpha\iota_0\alpha\iota_1^{-1})(\beta\iota_0\beta\iota_1^{-1})=\eta(\alpha)\eta(\beta).$$

Therefore, η is a homomorphism. Clearly, η induces a congruence equivalence relation $\sim$ in $Z^1(M,A)$. In other words, for $\alpha,\beta\in Z^1(M,A)$, the equivalence relation $\sim$ is defined as follows:
$\alpha\sim\beta$ whenever there is $\gamma\in Der_c(P,A)$ and $\beta=\gamma\iota_0\gamma\iota_1^{-1}\alpha$.

Definition 2.7. *It is said that a short exact sequence*

$$e:1\to A\xrightarrow{\chi}E\xrightarrow{\pi}G\to 1$$

is a proper extension of G by A whenever χ is a homeomorphic embedding and π is an open continuous homomorphism. By a section for e we mean a continuous map $s:G\to E$ such that $\pi s=Id_G$. Let e be a proper extension of G by A which has a section $s:G\to E$. In addition, assume that A is abelian. Then the proper extension e gives rise to a topological G-module structure on A, which is well-defined by
${}^g a=\chi^{-1}(s(g)\chi(a)s(g)^{-1})$, where $g\in G$, $a\in A$.

One can easily see that for each $g\in G$ and $a\in A$, the element ${}^g a$ dose not depend on the choice of continuous sections. Now, let A be an abelian topological G-module. We denote by $opext_s(G,A)$ the set of all proper extensions of G by A having continuous sections and corresponding to the given way in which G acts on A. We define an equivalence relation $\equiv$ on $opext_s(G,A)$ as follows: Let $e_i:1\to A\xrightarrow{\chi_i}E\xrightarrow{\pi_i}G\to 1\in opext_s(G,A)$ for $i=0,1$. $e_0\equiv e_1$ whenever there is a continuous (open) homomorphism $\theta:E_0\to E_1$ so that the following diagram is commutative.

$$\begin{array}{ccccccccc}
1\to & A & \xrightarrow{\chi_0} & E_0 & \xrightarrow{\pi_0} & G & \to 1\\
 & \| & & \downarrow^{\theta} & & \| & \\
1\to & A & \xrightarrow{\chi_1} & E_1 & \xrightarrow{\pi 1} & G & \to 1
\end{array}$$

We denote the set of all equivalence classes by $Opext_s(G,A)$. Note that if G is Hausdorff, then $Opext_s(G,A)=Ext_s(G,A)$. One can verify by a similar argument as in [1] that we have the following.

Theorem 2.8. *Let A be an abelian topological G-module, then $H^2(G,A)$ in the sense of Hu, is isomorphic to $Opext_s(G,A)$ with Baer sum.*

THE MAIN THEOREM

In this section we will prove the main theorem. Let A be a topological G-module. Thus, under the action $(g,a)\overset{\phi}{\longmapsto}{}^g a$, we can consider the topological semidirect product $A\rtimes_\phi G$ (see [7, Section 6]). We use for simplicity from $A\rtimes G$ instead of $A\rtimes_\phi G$ in the following.

Theorem 3.1. *Let A be an abelian topological G-module and let (M,ι_0,ι_1,τ) be a standard projective simplicial kernel of G. Then, $H^2(G,A)$ is canonically isomorphic to $Z^1(M,A)/\sim$.* *Proof.* By Theorem 2.8, it is enough to show that $Opext_s(G,A)$ is isomorphic to $Z^1(M,A)/\sim$. Let $\alpha\in Z^1(M,A)$. Since A is a G-module, then A is a P-module via τ. Thus, we may consider topological semidirect product $A\rtimes P$. Now, we define a relation $\sim$ in the topological semidirect product $A\rtimes P$, as follows:
$(a,x)\sim(b,y)$ if and only if $\tau(x)=\tau(y)$ and $a\alpha(x,y)=b$.

Obviously, $\sim$ is reflexive. Let x,y and z be arbitrary elements of P such that $\tau(x)=\tau(y)=\tau(z)$. Then

$$\alpha(x,y)\alpha(y,z)=\alpha(x,z) \qquad (3.1)$$

since,

$\alpha(x,z)=\alpha((x,y)(1,y^{-1}z))=$
$\alpha(x,y).^{y}\alpha(1,y^{-1}z)=\alpha(x,y).^{y}\alpha(y^{-1}y,y^{-1}z)$
$=\alpha(x,y).^{y}\alpha((y^{-1},y^{-1})(y,z))$
$=\alpha(x,y).^{y}(\alpha(y^{-1},y^{-1}).^{y^{-1}}\alpha(y,z))$
$=\alpha(x,y).^{y}(1.^{y^{-1}}\alpha(y,z))=\alpha(x,y)\alpha(y,z).$

Now if $(a,x)\sim(b,y)$ and $(b,y)\sim(c,z)$, then
$\tau(x)=\tau(y)=\tau(z)$, $a\alpha(x,y)=b$ and $b\alpha(y,z)=c$.
Hence, by (3.1), $a\alpha(x,z)=a\alpha(x,y)\alpha(y,z)=b\alpha(y,z)=c$. This implies that $\sim$ is transitive. By (3.1), we have:

$$\alpha^{-1}(x,y)=\alpha(y,x). \qquad (3.2)$$

If $(a,x)\sim(b,y)$, then $\tau(x)=\tau(y)$ and $a\alpha(x,y)=b$. Thus, $a=b\alpha^{-1}(x,y)=b\alpha(y,x)$. Consequently, $\sim$ is an equivalence relation in $A\rtimes P$. Now, Define $N_\alpha=\{(a,x)\,|\,(a,x)\in A\rtimes P,(a,x)\sim(1,1)\}$. Then, by the definition of $\sim$ and the identity (3.2), we have:
$N_\alpha=\{(a,x)\,|\,(a,x)\in A\rtimes P,\tau(x)=1,a\alpha(x,1)=1\}$
$=\{(a,x)\,|\,(a,x)\in A\rtimes P,\tau(x)=1,a=\alpha(1,x)\}.$

Suppose that $(a,x),(b,x)\in N_\alpha$. Since $\tau(x)=1$, then $(a,x)(b,y)=(a^{x}b,xy)=(ab,xy)$. On the other hand, $\alpha(1,xy)=\alpha(1,x)\alpha(1,y)=ab$. Thus, $(a,x)(b,y)\in N_\alpha$. Also $(a,x)^{-1}=({}^{x^{-1}}a^{-1},x^{-1})=(a^{-1},x^{-1})$ and
$\alpha(1,x^{-1})=\alpha(xx^{-1},x^{-1})=\alpha(x,1)^{x}\alpha(x^{-1},x^{-1})$
$=\alpha(x,1)=\alpha^{-1}(1,x)=a^{-1}$, hence $(a,x)^{-1}\in N_\alpha$. Therefore N_α is a subgroup of $A\rtimes P$.

Denote by E_α the quotient space $(A\rtimes P)/\sim$. We show that $\sim$ is a congruence (i.e., $\sim$ is compatible with the group product) and therefore E_α is a group.
Let $(a,x)\sim(a',x')$ and $(b,y)=(b',y')$. Then $\tau(x)=\tau(x')$, $\tau(y)=\tau(y')$, $a\alpha(x,x')=a'$ and $b\alpha(y,y')=b'$.
We have $(a,x)(b,y)=(a^{x}b,xy)$, $(a',x')(b',y')=(a'^{x'}b',x'y')$. Since A is abelian, then
$a^{x}b\alpha(xy,x'y')=a^{x}b\alpha(x,x')^{x}\alpha(y,y')$
$=a\alpha(x,x')^{x}(b\alpha(y,y'))=a'^{x}b'=a'^{x'}b'.$
Hence, $(a,x)(b,y)\sim(a',x')(b',y')$. Thus, E_α is a group. It is clear that $E_\alpha=(A\rtimes P)/N_\alpha$. This implies that N_α is a normal subgroup of $A\rtimes P$. Therefore E_α is a topological group.
There is a diagram as follows:

$$\begin{array}{ccccc}
M & \overset{t_0}{\underset{t_1}{\rtimes}} P & \xrightarrow{\tau} & G & \to 1\\
\downarrow^{\alpha} & \downarrow^{\beta} & & \| & \\
A & \xrightarrow{\sigma} E_\alpha & \xrightarrow{\psi} & G & \to 1
\end{array}$$

where $\sigma(a)=(a,1)N_\alpha$, $\psi(a,x)N_\alpha=\tau(x)$ and $\beta(x)=(1,x)N_\alpha$. It is obvious that σ and β are continuous homomorphisms. Consider the continuous map $\phi:A\rtimes P\to P,\ (a,x)\mapsto\tau(x)$. We know that τ has a continuous section $s:G\to P$. Hence, $\beta\circ s$ is a continuous section for ψ, since $\psi\beta\circ s=(\psi\beta)\circ s=\tau\circ s=Id_G$. Thus, by Lemma 2.6, ϕ is an open epimorphism. If $\sigma(a)\in N_\alpha$, then $(a,1)\in N_\alpha$. Thus, by definition of N_α, we get $a=\alpha(1,1)=1$. Therefore σ is one to one. $\psi\circ\sigma(a)=\psi((a,1)N_\alpha)=\tau(1)=1$, thus $Im\sigma\subset Ker\psi$. On the other hand, if $\psi(a,x)N_\alpha=1$, then $\tau(x)=1$ and $\sigma(a\alpha(x,1))=(a,x)N_\alpha$, i.e., $Ker\psi\subset Im\sigma$. Therefore, $Im\sigma=Ker\psi$. Now, we prove that σ is a homeomorphic embedding. Define the map $\chi : A ã P \to A$ via $\chi(a,x) = a\alpha(x, s\tau(x)s(1)^{-1})$. Obviously, χ is continuous. It is clear that $(A\times\{1\})N_\alpha$ is a topological subgroup of $A ã P$ and $(A\times\{1\})N_\alpha = \{(a,x) \mid a \in A, x \in Ker\tau\}$. Take $\overline{\chi} = \chi|_{(A\times\{1\})N_\alpha}$. Thus, $\overline{\chi}$ is continuous and $\overline{\chi}(a,x) = a\alpha(x,1)$. Note that $\overline{\chi}$ is a homomorphism, because $\overline{\chi}((a,x)(b,y)) = \overline{\chi}(ab,xy) = ab\alpha(xy,1) = ab\alpha(x,1)\alpha(xy,x) = ab\alpha(x,1)\alpha(x^{-1}xy,1) = ab\alpha(x,1)\alpha(y,1) = \overline{\chi}(a,x)\overline{\chi}(b,y)$. If $(a,x) \in N_\alpha$, then $\overline{\chi}(a,x) = a\alpha(x,1) = \alpha(1,x)\alpha(x,1) = \alpha(1,1) = 1$. Hence $\overline{\chi}(N_\alpha) = 1$. So, $\overline{\chi}$ induces the continuous homomorphism

$\xi : ((A\times\{1\})N_\alpha)/N_\alpha \to A, \xi((a,x)N_\alpha) = \overline{\chi}(a,x)$.

It is clear that $Im\,\sigma = ((A\times\{1\})N_\alpha)/N_\alpha$. We have $\xi\circ\sigma(a) = a$ and

$$\begin{aligned}\sigma\circ\xi((a,x)N_\alpha) &= \sigma(a\alpha(x,1))\\ &= (a\alpha(x,1),1)N_\alpha = (a,x)(\alpha(x,1),x^{-1})N_\alpha\\ &= (a,x)(\alpha(1,x^{-1}),x^{-1})N_\alpha = (a,x)N_\alpha.\end{aligned}$$

Thus, $\sigma|^{Im\sigma} : A \to Im\,\sigma$, $\sigma|^{Im\sigma}(a) = (a,1)N_\alpha$, is a topological isomorphism. Therefore, σ is a homeomorphic embedding.

Denote by e_α the extension $0 \to A \xrightarrow{\sigma} E_\alpha \xrightarrow{\psi} G \to 1$. We have $[e_\alpha] \in Opext_s(G,A)$. Since $s : G \to P$ is a continuous section for τ, then $\beta\circ s$ is a continuous section for ψ and $\beta\circ s(g)\sigma(a)\beta\circ s(g)^{-1} = (1,s(g))N_\alpha(a,1)N_\alpha(1,s(g)^{-1})N_\alpha = (1,s(g))(a,1)(1,s(g)^{-1})N_\alpha = ({}^{s(g)}a, s(g))(1,s(g)^{-1})N_\alpha$
$= ({}^{s(g)}a,1)N_\alpha = ({}^{g}a,1)N_\alpha = \sigma({}^{g}a)$.

In addition, σ is one to one, thus the extension e_α induces the given action of G on A. Hence $[e_\alpha] \in Opext_s(G,A)$. Consider the map $\zeta : Z^1(M,A) \to Opext_s(G,A)$ via $\alpha \mapsto [e_\alpha]$. Suppose that $\alpha, \overline{\alpha} \in Z^1(M,A)$ and $\overline{\alpha} = \eta(\gamma)\alpha$, for some $\gamma \in Der_c(P,A)$. Define the map $\varepsilon : A ã P \to A ã P$ by $(a,x) \mapsto (a\gamma^{-1}(x),x)$. Obviously, ε is continuous, and

$$\begin{aligned}&\varepsilon((a,x)(b,y)) = \varepsilon(a^x b, xy) = (a^x b\gamma^{-1}(xy), xy) =\\ &(a^x b^x \gamma^{-1}(y)\gamma^{-1}(x), xy) = (a\gamma^{-1}(x)^x(b\gamma^{-1}(y)), xy)\\ &= (a\gamma^{-1}(x),x)(b\gamma^{-1}(y),y);\end{aligned}$$

that is, ε is a homomorphism.

If $(a,x) \in N_\alpha$, then

$$\begin{aligned}&\overline{\alpha}(1,x) = (\gamma\iota_0.\gamma\iota_1^{-1}.\alpha)(1,x)\\ &= \gamma\iota_0(1,x)\gamma\iota_1^{-1}(1,x)\alpha(1,x)\\ &= \gamma^{-1}(x)\alpha(1,x) = a\gamma^{-1}(x).\end{aligned}$$

Hence, $(a\gamma^{-1}(x),x) \in N_{\overline{\alpha}}$, i.e., $\varepsilon(N_\alpha) \subset N_{\overline{\alpha}}$. Thus, ε induces the continuous homomorphism

$\overline{\varepsilon} : E_\alpha \to E_{\overline{\alpha}},\ (a,x)N_\alpha \mapsto (a\gamma^{-1}(x),x)N_{\overline{\alpha}}$.

Suppose that the extensions $e_\alpha : 0 \to A \xrightarrow{\sigma} E_\alpha \xrightarrow{\psi} G \to 1$ and $e_{\overline{\alpha}} : 0 \to A \xrightarrow{\overline{\sigma}} E_{\overline{\alpha}} \xrightarrow{\overline{\psi}} G \to 1$ are corresponding to α and $\overline{\alpha}$, respectively. We show that the following diagram commutes.

$$\begin{array}{ccccccccc} e_\alpha : 0 \to & A & \xrightarrow{\sigma} & E_\alpha & \xrightarrow{\psi} & G & \to 1\\ & \| & & \downarrow{\overline{\varepsilon}} & & \| & \\ e_{\overline{\alpha}} : 0 \to & A & \xrightarrow{\overline{\sigma}} & E_{\overline{\alpha}} & \xrightarrow{\overline{\psi}} & G & \to 1\end{array}$$

Because,

$$\overline{\psi}\overline{\varepsilon}(a,x)N_\alpha = \overline{\psi}(a\gamma^{-1}(x),x)N_{\overline{\alpha}} = \tau(x) = \psi(a,x)N_\alpha,$$

and

$$\overline{\varepsilon}\sigma(a) = \overline{\varepsilon}(a,1)N_\alpha = (a\gamma^{-1}(1),1)N_{\overline{\alpha}} = (a,1)N_{\overline{\alpha}} = \overline{\sigma}(a).$$

i.e., $[e_\alpha]=[e_{\bar{\alpha}}]$. Thus, ζ induces the map
$\Theta: Coker\eta \to Opext_s(G,A)$
$[\alpha]\mapsto [e_\alpha]$.

We will prove that Θ is a bijective map. Let $e_\alpha: 0\to A \xrightarrow{\sigma} C \xrightarrow{\psi} G \to 1$ be an extension with a continuous section $s: G\to A$ and corresponding to the given way in which G acts on A. For simplicity, we assume that σ is the inclusion map. Since P is a projective topological group, then there is a continuous homomorphism β so that

$$\begin{array}{ccc} P & \xrightarrow{\tau} & G \to 1 \\ \downarrow^{\beta} & & \| \\ C & \xrightarrow{\psi} & G \to 1 \end{array} \qquad 1$$

commutes. Now, define $\alpha_\beta : M\to A$ by $\alpha_\beta(m)=(\beta\iota_0\beta\iota_1^{-1})(m)$. Note that α_β is well-defined, because $\psi(\beta\iota_0\beta\iota_1^{-1})=\psi\beta\iota_0\psi\beta\iota_1^{-1}=\tau\iota_0\tau\iota_1^{-1}=1$. Thus, $Im\,\alpha_\beta\subset Ker\psi = Im\,\sigma = A$. For simplicity, we denote α_β by $\beta\iota_0\beta\iota_1^{-1}$. It is clear that α_β is continuous and in addition, α_β is a crossed homomorphism, since

$$\alpha_\beta((x_1,y_1)(x_2,y_2))=\alpha_\beta(x_1x_2,y_1y_2)=\beta(x_1x_2)\beta^{-1}(y_1y_2)=\beta(x_1)\beta(x_2)\beta^{-1}(y_2)\beta^{-1}(y_1)=\beta(x_1)\alpha_\beta(x_2,y_2)\beta^{-1}(y_1)$$
$$=\alpha_\beta(x_1,y_1)\beta(y_1)\alpha_\beta(x_2,y_2)\beta^{-1}(y_1)=\alpha_\beta(x_1,x_2).^{\beta(y_1)}\alpha_\beta(x_2,y_2))=\alpha_\beta(x_1,y_1).^{\psi\beta(y_1)}\alpha_\beta(x_2,y_2)$$
$$=\alpha_\beta(x_1,y_1).^{\tau(y_1)}\alpha_\beta(x_2,y_2)=\alpha_\beta(x_1,y_1).^{(x_1,y_1)}\alpha_\beta(x_2,y_2),$$

and $\alpha_\beta(x,x)=\beta(x)\beta^{-1}(x)=1$. i.e., $\alpha_\beta(\Delta_P)=1$, hence $\alpha_\beta\in Z^1(M,A)$. Consequently, β and α_β make the following diagram commutative.

$$\begin{array}{ccccc} M & \overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}} & P & \xrightarrow{\tau} & G\to 1 \\ \downarrow^{\alpha_\beta} & & \downarrow^{\beta} & & \| \\ 0\to A & \xrightarrow{\sigma} & E_\alpha & \xrightarrow{\psi} & G\to 1 \end{array}$$

Now, we will show that $[\alpha_\beta]$ is independent of the choice of $\beta: P\to C$. Consider the commutative diagram.

$$\begin{array}{ccccc} M & \overset{\iota_0}{\underset{\iota_1}{\rightrightarrows}} & P & \xrightarrow{\tau} & G\to 1 \\ \alpha_{\bar\beta}\downarrow^{\alpha_\beta} & & \bar\beta\downarrow^{\beta} & & \| \\ 0\to A & \xrightarrow{\sigma} & E_\alpha & \xrightarrow{\psi} & G\to 1 \end{array}$$

Define $\gamma(x)=(\bar\beta\beta^{-1})(x)$, for all $x\in P$. Since $\psi\gamma(x)=\psi\bar\beta(x)\psi\beta^{-1}(x)=\tau(x)\tau(x)^{-1}=1$, then $\gamma(x)\in Ker\psi = A$. Hence, we can define the continuous map $\gamma: P\to A$, $x\mapsto\gamma(x)$. Now γ is a crossed homomorphism, since

$$\gamma(xy)=(\bar\beta\beta^{-1})(xy)=\bar\beta(xy)\beta^{-1}(xy)$$
$$=\bar\beta(x)\bar\beta(y)\beta^{-1}(y)\beta^{-1}(x)$$
$$=\bar\beta(x)\gamma(y)\beta^{-1}(x)=\gamma(x)\beta(x)\gamma(y)\beta^{-1}(x)$$
$$=\gamma(x)^{\beta(x)}\gamma(y)=\gamma(x)^{\psi\beta(x)}\gamma(y)=\gamma(x)^{x}\gamma(y).$$

We have $\bar\beta=\gamma\beta$. Thus,

$$\alpha_{\bar\beta}=\bar\beta\iota_0\bar\beta\iota_1^{-1}=\gamma\beta\iota_0\gamma\beta\iota_0^{-1}=\gamma\iota_0\beta\iota_0\beta\iota_1^{-1}\gamma\iota_1^{-1}$$
$$=\gamma\iota_0(\beta\iota_0\beta\iota_1^{-1})\gamma\iota_1^{-1}=\gamma\iota_0\alpha_\beta\gamma\iota_1^{-1}=\alpha_\beta(\gamma\iota_0\gamma_1^{-1}).$$

i.e., $\alpha_{\bar\beta}\,\square\,\alpha_\beta$.

Now, we will show that $\Theta([\alpha_\beta]) = [e]$. It is sufficient to prove that e_{α_β} □ e. Define the map $\nu : A$ ã $P \to C$ via $(a,x) \mapsto a\beta(x)$. Obviously, ν is continuous, and since

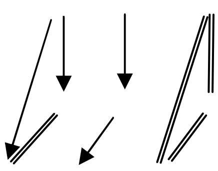
$$\nu((a,x)(b,y)) = \nu(a^x b, xy) = a^x b\,\beta(xy)$$
$$= a\beta(x)^{\beta(x)^{-1}}({}^x b)\beta(y) = \nu(a,x)^{\psi\beta(x)^{-1}}({}^x b)\beta(y)$$
$$= \nu(a,x)^{x^{-1}}({}^x b)\beta(y) = \nu(a,x)\nu(b,y),$$

then ν is a homomorphism. On the other hand, if $(a,x) \in N_{\alpha_\beta}$, then
$\nu(a,x) = a\beta(x) = \alpha_\beta(1,x)\beta(x) = \beta^{-1}(x)\beta(x) = 1.$

Thus, $\nu : A$ ã $P \to C$ induces the continuous homomorphism $\bar{\nu} : E_{\alpha_\beta} \to C$, $(a,x)N_{\alpha_\beta} \mapsto a\beta(x)$. Note that $\bar{\nu}$ commutes the following diagram,

$$\begin{array}{lccccccccc} e_{\alpha_\beta}: & 0 \to A & \xrightarrow{\bar{\sigma}} & E_{\alpha_\beta} & \xrightarrow{\bar{\psi}} & G \to 1 \\ & \| & & \downarrow \bar{\nu} & & \| \\ e\ : & 0 \to A & \xrightarrow{\sigma} & C & \xrightarrow{\psi} & G \to 1 \end{array}$$

Since, $\bar{\nu}\bar{\sigma}(a) = \bar{\nu}((a,1)N_{\alpha_\beta}) = a\beta(1) = a = \sigma(a)$, and $\psi\bar{\nu}((a,x)N_{\alpha_\beta}) = \psi(a\beta(x)) = \psi\beta(x) = \tau(x) = \bar{\psi}((a,x)N_{\alpha_\beta})$ Therefore, e_{α_β} □ e. i.e., Θ is onto. We show that Θ is one to one. Let $e_i : 0 \to A \xrightarrow{\sigma_i} C_i \xrightarrow{\psi_i} G \to 1$, $i = 0,1$, be the extensions with continuous sections and corresponding to the given way in which G acts on A. Let σ_i 's be the inclusion maps. Suppose that e_1 □ e_2 are equivalent, i.e., there is continuous homomorphism $\omega : C_1 \to C_2$ such that $\omega(a) = a, \forall a \in A$, and $\psi_2\omega = \psi_1$. There is a continuous homomorphism $\beta : P \to C_1$ such that $\psi_1\beta = \tau$. Take $\bar{\beta} = \omega \circ \beta$. Consider the following diagram.



M Ã $P \xrightarrow{\tau} G \to 1$ (t_0, t_1)

$\alpha_\beta \quad \alpha_{\bar{\beta}} \quad \beta$

$0 \to A \xrightarrow{\sigma_1} C_1 \overset{\psi_1}{- - \to} G \to 1$

ω

$0 \to A \underset{\sigma_2}{\to} C_1 \underset{\psi_2}{- - \to} G \to 1$

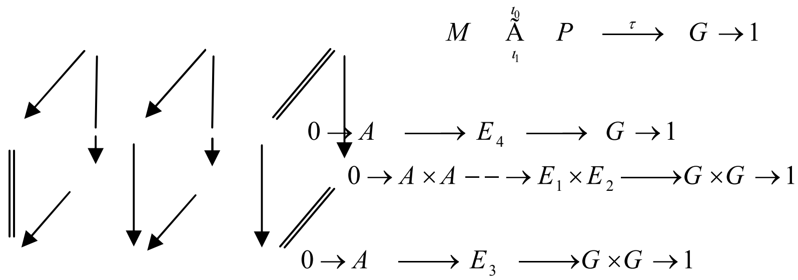
Since the back, the bottom, the front squares, and the middle triangle are commutative, so is the left triangle, i.e., $\alpha_{\bar{\beta}} = \alpha_\beta$. This shows that Θ is one to one. Consequently, Θ is bijective. Finally, we prove that Θ is a homomorphism. Suppose that $\Theta([\alpha_i]) = [e_i]$, $i = 0,1$. In another words,the following diagram is commutative

$$\begin{array}{lccc} M \overset{t_0}{\underset{t_1}{\text{Ã}}} & P \xrightarrow{\tau} & G \to 1 \\ \downarrow \alpha_i & \downarrow \beta_i & \| \\ e_i : 0 \to A \xrightarrow{\sigma_i} & E_i \xrightarrow{\psi_i} & G \to 1 \end{array}$$

for $i = 0,1$. Hence, we have the commutative diagram

$$\begin{array}{lccc} M \overset{t_0}{\underset{t_1}{\text{Ã}}} & P \xrightarrow{\tau} & G \to 1 \\ \alpha_1\times\alpha_2 \downarrow & \downarrow \beta_1\times\beta_2 & \downarrow \Delta_G \\ e_1 \times e_2 : 0 \to A \times A \xrightarrow{\sigma_1\times\sigma_2} & E_1 \times E_2 \xrightarrow{\psi_1\times\psi_2} & G \times G \to 1 \end{array}$$

where $\Delta_G(g) = (g,g)$, $\beta_1 \times \beta_2(p) = (\beta_1(p), \beta_2(p))$, $\alpha_1 \times \alpha_2(m) = (\alpha_1(m), \alpha_2(m))$, $\sigma_1 \times \sigma_2(a,b) = (\sigma_1(a), \sigma_2(b))$ and $\psi_1 \times \psi_2(e_1, e_2) = (\psi_1(e_1), \psi_2(e_2))$ .Then by definition of pushout and pullback extensions[1], we get the following diagram



in which, the arrow $G \to G \times G$ is the diagonal map, the arrow $A \times A \to A$ is the codiagonal map ∇_A , i.e., $\nabla_A(a,b) = ab$, and the arrow $M \to A$ is the multiplication map $\alpha_1.\alpha_2$ by the rule $\alpha_1.\alpha_2(m) = \alpha_1(m)\alpha_2(m)$. The extension $0 \to A \to E_3 \to G \times G \to 1$ is the pushout extension of $e_1 \times e_2$, and extension $0 \to A \to E_4 \to G \to 0$ is the pullback extension of $\nabla_A(e_1 \times e_2)$. Obviously, the left and the right squares are commutative. Since P is a projective topological group, then the existence of $\gamma : P \to E_4$ is guaranteed and commutes the right-up square. Therefore in the right cube of the above diagram, all the up, down, front, back and right squares are commutative. So γ commutes the middle square. On the other hand, in the left cube of the above diagram, all the right, left, back, front and down squares are commutative. Thus, γ commutes the left-up square of the above diagram. Therefore, the whole top of the diagram is commutative. This means that $\Theta([\alpha_1.\alpha_2]) = [e_4]$. On the other hand by definition of Baer sum in the $Opext_s(G,A)$, $[\nabla_A(e_1 \times_2)\Delta_G] = [e_1] + [e_2]$. Consequently, $\Theta(\alpha_1.\alpha_2) = [e_1] + [e_2] = \Theta(\alpha_1) + \Theta(\alpha_2)$, i.e., Θ is an isomorphism and thereby Θ is an isomorphism. The proof is completed. □ Note that Theorem 3.1 is similar to [5, Theorem 6] in a topological context. The following corollary is a general result of [4, (5.6)].

Corollary 3.2. *Let P be a projective topological group, then $H^2(P,A) = 0$, for every abelian topological P -module A . In paricular, for every Markov (Graev) free topological groups F , $H^2(F,A) = 0$, for every abelian topological F -module A .* *Proof.* It is clear that $(\Delta_P, t_0, t_G, Id_P)$ is a standard projective simplicial kernel of P . Thus, every continuous crossed homomorphism $\alpha : \Delta_P \to A$ is the zero map. Consequently, $H^2(P,A) = 0$. Now by Lemma 2.2, for every Markov (Graev) free topological group F , $H^2(F,A) = 0$.□

REFERENCES

1. R.C. Alperin and H. Sahleh, Hopf's formula and the Schur multiplicator for topological groups, Kyungpook. Math. Journal, 31(1), 35-71, (1991).
2. M. I. Graev, Free topological groups, (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. 12, 279-324, (1948).
3. C. E. Hall, F -Projective Objects, Proc. Amer. Math. Soc. 26, 193-195, (1970).
4. S.T. Hu, Cohomology theory in topological groups, Michigan Math. J., 1(1), 11-59, (1952).
5. H. Inassaridze, Non-abelian cohomology of groups, Georgian Math. J. 4, No 4, 313-332, (1997).
6. A. A. Markov, On free topological groups. (Russian) Izv. Akad. Nauk SSSR, Ser. Mat. 9, 3-64, (1945).
7. H. Sahleh, H.E. Koshkoshi, First Non-Abelian Cohomology Of Topological Groups. Gen. Math. Notes, 12(1), 20-34, (2012).
8. B.V.S. Thomas, Free topological groups, General Topology and its Appl., 1, pp. 51-72, (1971).

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