Research Article

Some common fixed point theorems for sequence of mappings in two M-fuzzy metric spaces

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Abstract In this paper we prove some common fixed point theorems for sequence of mappings in two complete M-fuzzy metric spaces.

Key words and Phrases*:* fixed point, common fixed point and complete M-fuzzy metric space.
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INTRODUCTION

After introduction of fuzzy sets by Zadeh[5], Kramosil and Michalek[4] introduced the concept of fuzzy metric space in 1975. Consequently in due course of time many researchers have defined a fuzzy metric space in different ways. Researchers like George and Veeramani[1], Grabiec[6], Subrahmanyam[8] used this concept to generalize some metric fixed point results. Recently, Sedghi and Shobe[9] introduced M-fuzzy metric space which is based on D*-metric concept. In this paper, we prove some common fixed point theorems for sequence of mappings in two M-fuzzy metric spaces. First we give some known definitions and results in M-fuzzy metric space and then prove our main result.

Definition: 1.1[9]: Let X be a nonempty set. A D' - metric (or generalized metric) on X is a function: $D': X^3 \to [0, \infty)$, that satisfies the following conditions for each x, y, z, a $\in$ X

(i) $D'(x, y, z) \geq 0$,

(ii) $D'(x, y, z) = 0$ iff $x = y = z$,

(iii) $D'(x, y, z) = D'(p\{x, y, z\})$, (symmetry) where p is a permutation function,

(iv) $D'(x, y, z) \leq D'(x, y, a) + D'(a, z, z)$.

The pair (X, D, '), is called a generalized metric (or D' - metric) space.

Examples of D' - metric are

(a) $D'(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$,

(b) $D'(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

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Definition: 1.2: A fuzzy set M in an arbitrary set X is a function with domain X and values in [0, 1].
Definition: 1.3[2]: A binary operation *: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) is associative and commutative,
(ii) * is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each a, b, c, d $\in [0, 1]$.

Two typical examples for continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.
Definition: 1.4[9]: A 3-tuple (X, M, $*$) is called a M- fuzzy metric space. if X is an arbitrary non-empty set, * is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions for each x, y, z, a $\in$ X and t, s > 0
(FM – 1) M (x, y, z, t) > 0
(FM – 2) M (x, y, z, t) = 1 iff x = y = z
(FM – 3) M (x, y, z, t) = M (p {x, y, z}, t), where p is a permutation function
(FM – 4) M (x, y, a, t) * M (a, z, z, s) ≤ M (x, y, z, t+s)
(FM – 5) M (x, y, z, ·): $(0,\infty) \rightarrow [0,1]$ is continuous
(FM – 6) $\lim_{t \to \infty}$ M (x, y, z, t) = 1.
Example: 1.5: Let X be a nonempty set and D* is the D* - metric on X. Denote a*b = a. b for all a, b $\in$ [0, 1]. For each t $\in (0,\infty)$, define
$M(x, y, z, t) = t / (t+D^*(x,y.z))$
for all x, y, z $\in$ X, then (X, M, *) is a M- fuzzy metric space.
Lemma: 1.6: Let (X, M, $*$) be a M- fuzzy metric space. Then for every t > 0 and for every x, y $\in$ X we have M (x, x, y, t) = M(x, y, y,t).
Proof:
For each $\in$ > 0 by triangular inequality
We have

(i) $M(x, x, y, \in + t) \geq M(x, x, x, \in) * M(x, y, y, t)$
$= M(x, y, y, t)$
(ii) $M(y, y, x, \in + t) \geq M(y, y, y, \in) * M(y, x, x, t)$
$= M(y, x, x, t)$.

By taking limits of (i) and (ii) when $\in \rightarrow 0$,
we obtain M(x, x, y, t) = M(x, y, y, t)
Lemma: 1.7 Let (X, M, $*$) be a M- fuzzy metric space. Then M (x, y, z, t) is non-decreasing with respect to t, for all x, y, z in X.
Definition: 1.8 Let (X, M, $*$) be a M- fuzzy metric space. . For t > 0, the open ball B_M (x, r, t) with center x $\in$ X and radius 0 < r < 1 is defined by
$B_M(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}$.
A subset A of X is called open set if for each x $\in$ A there exist t > 0 and 0 < r < 1 such that
$B_M(x, r, t) \subseteq A$.
Definition: 1.9[9] Let (X, M, $*$) be a M- fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to converge to a point x $\in$ X if
$\lim_{n \to \infty} M(x, x, x_n, t) = 1$ for all t > 0
(b) $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all t > 0 and p > 0

Remark: 1.10 A M- fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
Definition: 1.11 A point x in X is said to be a common fixed point of sequence of maps T_n: X → X if $T_n(x) = x$ for all n.
Remark: 1.12 Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence is uniquely determined.
Lemma: 1.13[4] Let $\{x_n\}$ be a sequence in a M- fuzzy metric space. (X, M, $*$) with the condition (FM-6). If there exists a number q $\in$ (0, 1) such that
$M(x_n, x_n, x_{n+1}, t) \geq M(x_{n-1}, x_{n-1}, x_n, t/q)$
for all t > 0 and n = 1, 2, 3, …, then $\{x_n\}$ is a Cauchy sequence**.**
Lemma 1.14 [4] Let (X, M, $*$) be a M- fuzzy metric space with condition (FM-6). If for all x, y, z $\in$X, t > 0 with positive number q $\in$ (0,1) and M (x, y, z, qt) ≥ M (x, y, z, t), then x = y = z.

MAIN RESULTS

Theorem 2.1: Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces. If T_i is a mapping from X into Y and S_j is a mapping from Y into X satisfying

$$2M_1(S_jy,S_jy,S_jT_ix,qt) \geq M_1(x,x,S_jT_ix,t) + M_2(y,y,T_ix,t) \quad (1)$$

$$2M_2(T_ix,T_ix,T_iS_jy,qt) \geq M_2(y,y,T_iS_jy,t) + M_1(x,x,S_jy,t) \quad (2)$$

for all $i \neq j$ in N, x in X and y in Y where $q < 1$, then $\{S_nT_n\}$ has a unique common fixed point z in X and $\{T_nS_n\}$ has a unique common fixed point w in Y. Further $\{T_n\}z = w$ and $\{S_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows:

$x_n = (S_nT_n)^n x_0$, $y_n = T_n(x_{n-1})$

for n = 1,2,… . By (1) we have

$$\begin{aligned}2M_1(x_n, x_n ,x_{n+1},qt) &= 2M_1((S_nT_n)^n x_0,(S_nT_n)^n x_0,(S_nT_n)^{n+1} x_0),qt)\\ &= 2M_1(S_n(T_n(S_nT_n)^{n-1}x_0 , S_n(T_n (S_nT_n)^{n-1}x_0,\\ &\quad S_nT_n (S_nT_n)^n x_0 ,qt)\\ &= 2M_1(S_nT_n(x_{n-1}) , S_nT_n (x_{n-1}) , S_nT_nx_n, qt)\\ &= 2M_1(S_ny_n , S_ny_n , S_nT_nx_n ,qt)\\ &\geq M_1(x_n, x_n, S_nT_nx_n,t) + M_2(y_n, y_n,T_nx_n,t)\\ &= M_1(x_n , x_n, x_{n+1},t) + M_2(y_n, y_n, y_{n+1},t)\\ &\geq M_1(x_n , x_n, x_{n+1},qt) + M_2(y_n, y_n, y_{n+1},t)\end{aligned}$$

Thus we have

$$M_1(x_n, x_n,x_{n+1},qt) \geq M_2(y_n, y_n, y_{n+1},t) \quad (3)$$

Similarly, by (2)

$$\begin{aligned}2M_2(y_n, y_n,y_{n+1},qt) &= 2M_2(T_nx_{n-1}, T_nx_{n-1}, T_nx_n, qt)\\ &= 2M_2(T_nx_{n-1}, T_nx_{n-1}, T_nS_ny_n,qt)\\ &\geq M_2(y_n, y_n,T_nS_ny_n,t) + M_1(x_{n-1}, x_{n-1},S_ny_n,t)\\ &= M_2(y_n, y_n,y_{n+1},t)+ M_1(x_{n-1}, x_{n-1},x_n,t)\\ &\geq M_2(y_n, y_n,y_{n+1},qt)+ M_1(x_{n-1}, x_{n-1},x_n,t)\end{aligned}$$

Thus we have

$$M_2 (y_n, y_n, y_{n+1},qt) \geq M_1(x_{n-1}, x_{n-1}, x_n,t) \quad (4)$$

Therefore, by (3) and (4)

$$\begin{aligned}&M_1(x_n,x_n,x_{n+1},qt) \geq M_2(y_n,y_n,y_{n+1},t)\\ &\geq M_1(x_{n-1}, x_{n-1},x_n,t/q)\\ &\quad \vdots\\ &\geq M_1(x_0, x_0, x_1,t/q^n) \to 1 \text{ as } n\to\infty\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M_1,*)$ is complete, $\{x_n\}$ converges to a point z in X. Similarly we prove $\{y_n\}$ converges to a point w in Y. Again by (2) we have

$$\begin{aligned}2 M_2(T_nz,T_nz,y_{n+1},qt) &= M_2(T_nz,T_nz,T_nS_ny_n,qt)\\ &\geq M_2(y_n,y_n,T_nS_ny_n,t)+ M_1(z,z,S_ny_n,t)\\ &= M_2(y_n,y_n,y_{n+1},t) + M_1(z,z,x_n,t) \quad (5)\end{aligned}$$

Letting $n \to \infty$ in (5) we have

$2M_2(T_nz,T_nz,w,qt) \geq 2$

That is $M_2(T_nz,T_nz,w,qt) \geq 1$

which implies that $M_2(T_nz,T_nz,w,qt) =1$ so that $T_nz = w$.

On the other hand, by (1) we have

$$\begin{aligned}2M_1(S_nw, S_nw, x_{n+1},qt) &= 2M_1(S_nw,S_nw,S_nT_nx_n,t)\\ &\geq M_1(x_n,x_n,S_nT_nx_n,t)+M_2(w,w,T_nx_n,t)\\ &= M_1(x_n, x_n,x_{n+1},t)+M_2(w,w,y_{n+1},t) \quad (6)\end{aligned}$$

Letting $n\to\infty$ in (6), it follows that $S_nw = z$. Therefore we have $S_nT_nz = S_nw = z$ and $T_nS_nw = T_nz = w$ for all n, which means that the point z is a fixed point of S_nT_n and the point w is a fixed point of T_nS_n. To prove the uniqueness of the fixed point z, let z′ be the second fixed point of S_nT_n. By (1) we have

$2M_1(z ,z, z',qt) = 2M_1(S_n w, S_n w, S_nT_n z',qt)$
$\geq M_1(z',z',S_nT_nz',t)+M_2(w,w,T_nz',t)$
$\geq M_1(z',z',z',qt) + M_2(w,w,T_nz',t)$
Which implies that
$M_1(z ,z, z',qt) \geq M_2(w,w,T_nz',t)$ (7)
Similarly by (2), we have
$2M_2(w,w,T_nz',qt) = 2M_2(T_nz,T_nz,T_nS_nT_nz',qt)$
$\geq M_2(T_nz',T_nz',T_nS_nT_nz',t)+ M_1(z,z,S_nT_nz',t)$
$\geq M_2(T_nz',T_nz',T_nz',qt)+ M_1(z,z,z',t)$
Which implies that
$M_2(T_nz,T_nz,T_nz',qt) \geq M_1(z,z,z',t)$ (8)
Therefore by (7) and (8)
$M_1(z,z, z',qt) \geq M_2(T_nz, _nz,T_nz',t) \geq M_1(z,z, z',t/q)$ (since $q<1$),
which is a contradiction. Thus $z = z'$. So the point z is the unique fixed point of $\{S_nT_n\}$in X. Similarly, we prove the point w is also a unique fixed point of $\{T_nS_n\}$ in Y.

Remark: 2.2 In the above theorem 2.1, we have
$M_1(S_jy,S_jy,S_jT_ix,qt) \geq ½[M_1(x,x,S_jT_ix,t) + M_2(y,y,T_ix,t)]$
$\geq \min\{ M_1(x,x,S_jT_ix,t), M_2(y,y,T_ix,t) \}$
$M_2(T_ix,T_ix,T_iS_jy,qt) \geq ½[M_2(y,y,T_iS_jy,t) + M_1 (x,x,S_jy,t)]$
$\geq \min\{M_2(y,y,T_iS_jy,t),M_1 (x,x,S_jy,t)\}$
Hence we get the following corollary.

Corollary: 2.3 Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces. If T_i is a mapping from X into Y and S_j is a mapping from Y into X satisfying
$M_1 (S_jy,S_jy,S_jT_ix,qt) \geq \min\{M_1(x,x,S_jT_ix,t) , M_2(y,y,T_ix,t)\}$
$M_2(T_ix,T_ix, T_iS_jy,qt) \geq \min\{M_2(y,y,T_iS_jy,t) , M_1 (x,x,S_jy,t)\}$
for all $i \neq j$ in N, x in X and y in Y where $q < 1$, then $\{S_nT_n\}$ has a unique common fixed point z in X and $\{T_nS_n\}$has a unique common fixed point w in Y. Further $\{T_n\}z = w$ and $\{S_n\}w = z$.

Corollary: 2.4 Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying
$2M_1 (Sy,Sy,STx,qt) \geq M_1 (x,x,STx,t) + M_2(y,y,Tx,t)$
$2M_2(Tx, Tx, TSy,qt) \geq M_2(y,y,TSy,t) + M_1(x,x,Sy,t)$
for all x in X and y in Y where $q < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Theorem 2.5: Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces. Let A_i, B_j be mappings of X into Y and S_p, T_q be mappings of Y into X satisfying the inequalities.
$3M_1((S_pA_ix,T_qB_jx',T_qB_jx',qt) \geq M_1(x,x',x',t)+M_1(x, x,S_pA_ix ,t) +M_1(x', x' ,T_qB_jx',t)$ (1)
$3M_2(B_jS_py,A_iT_qy',A_iT_qy',qt) \geq M_2(y.y',y',t)+M_2(y, y, B_jS_py ,t) + M_2(y', y',A_iT_q y',t)$ (2)
for all $i \neq j \neq p \neq q$ in N, x, x′ in X and y,y′ in Y where $q<1$. If one of the mappings $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous, then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w =\{ T_n \}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by
$A_nx_{2n-2} = y_{2n-1}, S_ny_{2n-1} = x_{2n-1}, B_nx_{2n-1} = y_{2n}; T_ny_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .
Now we have
$3M_1(x_{2n+1}, x_{2n}, x_{2n},qt) = 3M_1(S_nA_nx_{2n}, T_nB_nx_{2n-1}, T_nB_nx_{2n-1},qt)$
$\geq M_1(x_{2n},x_{2n-1},x_{2n-1},t)+M_1(x_{2n},x_{2n}, S_nA_nx_{2n},t) + M_1(x_{2n-1}, x_{2n-1}, T_nB_nx_{2n-1},t)$

$= M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) + M_1(x_{2n}, x_{2n}, x_{2n+1},t) +$
$M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)$
$=2 M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) + M_1(x_{2n}, x_{2n}, x_{2n+1}, t)$
$\geq 2 M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) + M_1(x_{2n+1}, x_{2n}, x_{2n}, qt)$

Which implies that
$2M_1(x_{2n+1}, x_{2n}, x_{2n},qt) \geq 2 M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t)$
That is $M_1(x_{2n+1},x_{2n},x_{2n},qt) \geq M_1(x_{2n},x_{2n-1},x_{2n-1},t)$ (3)
Again using (1)
$3M_1(x_{2n}, x_{2n-1}, x_{2n-1},qt) = 3M_1(x_{2n-1}, x_{2n}, x_{2n},qt)$
$= 3M_1(S_nA_nx_{2n-2}, T_nB_nx_{2n-1}, T_nB_nx_{2n-1},qt)$
$\geq M_1(x_{2n-2},x_{2n-1},x_{2n-1},t)+$
$M_1(x_{2n-2},x_{2n-2}, S_nA_nx_{2n-2}, t) +$
$M_1(x_{2n-1}, x_{2n-1},T_nB_nx_{2n-1},t)$
$= M_1(x_{2n-2},x_{2n-1},x_{2n-1},t)+M_1(x_{2n-2},x_{2n-2},x_{2n-1},t) +$
$M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)$
$\geq 2M_1(x_{2n-2},x_{2n-1},x_{2n-1},t)+ M_1(x_{2n-1}, {}_{2n-1}, x_{2n}, qt)$

Which implies that
$2M_1(x_{2n},x_{2n-1}, x_{2n-1}, qt) \geq 2 M_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t)$
That is $M_1(x_{n+1},x_n,x_n,qt) \geq M_1(x_n,x_{n-1},x_{n-1},t)$ (4)
Thus from inequalities (3) and (4), we have

$$\begin{aligned} M_1(x_{n+1}, x_{2n}, x_{2n},qt) &\geq M_1(x_n, x_{n-1}, x_{n-1}, t) \\ &\geq M_1(x_{n-1}, x_{n-2}, x_{n-2}, t/q) \\ &\vdots \\ &\geq M_1(x_1, x_0, x_0, t/q^{n-1}) \to 1 \text{ as } n\to\infty \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since (X, M_1,*) is complete, $\{x_n\}$ converges to a point z in X. Similarly applying inequality (2) and proceeding as above, we prove $\{y_n\}$ converges to a point w in Y. Suppose $\{A_n\}$ is continuous, then
$\lim_{n\to\infty} A_nx_{2n} = A_nz = \lim_{n\to\infty} y_{2n+1} = w.$
Now we prove $S_nA_nz = z$ for all *n*.
We have

$$\begin{aligned} 3M_1(S_nA_nz,z, z,qt) &= \lim_{n\to\infty} 3M_1(S_nA_nz, T_nB_nx_{2n-1},T_nB_nx_{2n-1},qt) \\ &\geq \lim_{n\to\infty} \{ M_1 (z,x_{2n-1},x_{2n-1},t) + \\ &\quad M_1(z,z,S_nA_nz,t)+ \\ &\quad M_1(x_{2n-1},x_{2n\ 1},T_nB_nx_{2n-1}, t) \} \\ &= M_1(z,z,z,t)+M_1(z,z,S_nA_nz,t)+M_1(z,z,z,t) \\ &= 1+ M_1(z,z,S_nA_nz,t)+ 1 \\ &\geq 2 + M_1(z,z,S_nA_nz,qt) \end{aligned}$$

Which implies
$M_1(S_nA_nz, z, z,qt) \geq 1$
That is . $M_1(S_nA_nz, z, z,qt) = 1$
Thus $S_nA_nz = z$ for all *n*.
Hence $S_nw = z$ for all *n*. (Since $A_nz = w$ for all *n*.)
Now we prove $B_nS_nw = w$ for all *n*.
We have

$$\begin{aligned} 3 M_2(B_nS_nw, w,w,qt) &= \lim_{n\to\infty} 3 M_2(B_nS_nw, y_{2n+1}, y_{2n+1},qt) \\ &= \lim_{n\to\infty} 3M_2(B_nS_nw,A_nT_ny_{2n},A_nT_ny_{2n},qt) \\ &\geq \lim_{n\to\infty} \{ M_2(w, y_{2n}, y_{2n},t) + \end{aligned}$$

$M_2(w,w,B_nS_nw,t)+$
$M_2(y_{2n},y_{2n},A_nT_ny_{2n},t)\}$

Which implies $M_2(B_nS_nw, w,w,qt) = 1$.
Thus $B_nS_nw = w$ for all n.
Hence $B_nz = w$ for all n. (Since $S_nw = z$)
Now we prove $T_nB_nz = z$ for all n.

$$3M_1(z, T_nB_nz,T_nB_nz,qt) = \lim_{n\to\infty} 3M_1(x_{2n+1}, T_nB_nz,T_nB_nz,qt)$$
$$= \lim_{n\to\infty} 3M_1(SAx_{2n},T_nB_nz,T_nB_nz, qt)$$
$$\geq \lim_{n\to\infty} \{M_1(x_{2n},z,z,t) + M_1(x_{2n},x_{2n},S_nA_nx_{2n},t) + M_1(z, z,T_nB_nz,t)\}$$

Which implies $M_1(z,_{n\,,}T_nB_nz,T_nB_nz,qt) = 1$.
Thus $T_nB_nz = z$ for all n.
Hence $T_nw = z$ for all n. (Since $B_nz = w$)
Now we prove $A_nT_nw = w$ for all n.

$$3M_2(w, A_nT_nw, A_nT_nw,qt) = \lim_{n\to\infty} 3M_2(y_{2n}, A_nT_nw, A_nT_nw,qt)$$
$$= \lim_{n\to\infty} 3M_2(B_nS_ny_{2n-1}, A_nT_nw, A_nT_nw,qt)$$
$$\geq \lim_{n\to\infty} \{M_2(y_{2n-1},w,w,t) + M_2(y_{2n-1},y_{2n-1},B_nS_ny_{2n-1},t) + M_2(w,w,A_nT_nw,t)$$

Thus $A_nT_nw = w$ for all n.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

Uniqueness: Let z′ be another common fixed point of of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X, w′ be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

We have

$$3M_1(z, z', z',qt) = 3M_1(S_nA_nz, T_nB_nz', T_nB_nz',qt)$$
$$\geq M_1(z, z',z',t) + M_1(z ,z, S_nA_nz, t) + M_1(z', z', T_nB_nz', t),$$
$$= M_1(z, z', z',t) + M_1(z, z, z,t) + M_1(z',z', z',t)$$
$$\geq M_1(z, z', z',qt) + 2$$

Which implies $M_1(z, z', z',qt) \geq 1$

Thus $z = z'$. So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

Remark: 2.6 In the above theorem 2.5, we have

$$M_1((S_pA_ix,T_qB_jx', T_qB_jx',qt) \geq 1/3\{ M_1(x,x',x',t) + M_1(x,x,S_pA_ix,t)+ M_1(x', x' ,T_qB_jx',t)\}$$
$$\geq \min\{ M_1(x,x',x',t) , M_1(x,x,S_pA_ix,t), M_1(x', x' ,T_qB_jx',t)\}$$
$$M_2(B_jS_py, A_iT_qy', A_iT_qy',,qt) \geq 1/3\{ M_2(y.y',y',t) + M_2(y,y,B_jS_py,t)+ M_2(y',y',A_iT_qy',t)\}$$
$$\geq \min \{M_2(y.y',y',t) , M_2(y,y,B_jS_py,t), M_2(y',y',A_iT_qy',t)\}$$

Hence we get the following corollary.

Corollary: 2.7 Let (X, M_1, ✱) and (Y, M_2, ✱) be two complete M- fuzzy metric spaces. Let A_i, B_j be mappings of X into Y and S_p, T_q be mappings of Y into X satisfying the inequalities.

$$M_1(S_pA_ix, T_qB_jx', T_qB_jx',qt) \geq \min\{ M_1(x,x',x',t), M_1(x, x, S_pA_ix ,t) , M_1(x', x' ,T_qB_jx',t)\}$$

$$M_2(B_jS_py, A_iT_qy', A_iT_qy',,qt) \geq \min \{M_2(y.y',y',t), M_2(y, y, B_jS_py ,t) , M_2(y',y',A_iT_qy',t)\}$$

for all $i \neq j \neq p \neq q$ in N, x, x′ in X and y,y′ in Y where q<1. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w =\{ T_n \}w = z$.

Remark: 2.8 If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.1, we get the following corollary.

Corollary: 2.9 Let (X, M_1, ✱) and (Y, M_2, ✱) be two complete M- fuzzy metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$$3M (SAx, TBx',TBx',qt) \geq M_1(x,x',x',t) + M_1(x, x, SAx ,t) +M_1(x', x' ,TBx',t)$$

$$3M_2(BSy, ATy', ATy',qt) \geq M_2(y.y',y',t) + M_2(y, y, BSy ,t) + M_2(y', y', ATy',t)$$

for all x, x′ in X and y,y′ in Y where q<1. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Remark: 2.10 If (X, M_1,*) and (Y, M_2,*) are the same M- fuzzy metric spaces in the above theorem 2.1, then we obtain the following theorem as corollary.

Corollary: 2.11 [10] Let (X, M,*) be a complete M- fuzzy metric space and $T_n : X \to X$ be a sequence of maps such that for all t > 0 and 0 < k < 1 satisfying the condition

$$3M (T_ix, T_jy, T_jy, t) \geq \{ M(x, y, y, t/k) + M(x, x, T_ix, t/k) + M(y, y, T_jy,t/k)\}$$

for all $i \neq j$ and for all x, y in X. Then $\{T_n\}$ have a unique common fixed point.

Theorem 2.12: Let (X, M_1, ✱) and (Y, M_2, ✱) be two complete M- fuzzy metric spaces with continuous t-norm* defined by a*b = min{a,b}for all a, b $\in$[0,1]. If T_i is a mapping from X into Y and S_j is a mapping from Y into X satisfying

$$M_1(S_jy,S_jy,S_jT_ix,qt) \geq M_1(S_jy,S_jy,x,t) * M_1(x,x,S_jT_ix,t) * M_2(y,y,T_ix,t) \quad (1)$$

$$M_2(T_ix,T_ix,T_iS_jy,qt) \geq M_2(T_ix,T_ix,y,t) * M_2(y,y,T_iS_jy,t) * M_1(x,x,S_jy,t) \quad (2)$$

for all $i \neq j$ in N, x in X and y in Y where q < 1, then $\{S_nT_n\}$ has a unique common fixed point z in X and $\{T_nS_n\}$has a unique common fixed point w in Y. Further $\{T_n\}z = w$ and $\{S_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows:

$x_n = (S_nT_n)^n x_0$, $y_n = T_n(x_{n-1})$

for n = 1,2,… . By (1) we have

$$\begin{aligned} M_1(x_n ,x_n ,x_{n+1},qt) &= M_1((S_nT_n)^nx_0, (S_nT_n)^nx_0 , (S_nT_n)^{n+1} x_0),qt) \\ &= M_1(S_n(T_n (S_nT_n)^{n-1} x_0, S_n (T_n(S_nT_n)^{n-1}x_0 , S_nT_n (S_nT_n)^n x_0 ,qt) \\ &= M_1(S_nT_n(x_{n-1}) , S_nT_n(x_{n-1}) , S_nT_nx_n, qt) \\ &= M_1 (S_ny_n , S_ny_n , S_nT_nx_n ,qt) \\ &\geq M_1(S_ny_n,S_ny_n,x_n,t) * M_1(x_n,x_n,S_nT_nx_n,t) * M_2(y_n,y_n,T_nx_n,t) \\ &= M_1(x_n,x_n,x_n,t) * M_1(x_n,x_n,x_{n+1},t) * M_2(y_n,y_n,y_{n+1},t) \end{aligned}$$

$\geq 1 * M_1(x_n,x_n,x_{n+1},qt) * M_2(y_n,y_n,y_{n+1},t)$
$\geq M_2(y_n,y_n,y_{n+1},t)$

Thus we have, $M_1(x_n, x_n,x_{n+1},qt) \geq M_2(y_n, y_n, y_{n+1},t)$ (3)

Similarly, by (2)

$M_2(y_n, y_n,y_{n+1},qt) = M_2(T_nx_{n-1}, T_nx_{n-1}, T_nx_n, qt)$
$= M_2(T_nx_{n-1}, T_nx_{n-1}, T_nS_ny_n,qt)$
$\geq M_2(T_nx_{n-1},T_nx_{n-1},y_n,t) *$
$M_2(y_n,y_n,T_nS_ny_n,t) *$
$M_1(x_{n-1},x_{n-1},S_ny_n,t)$
$= M_2(y_n,y_n,y_n,t) * M_2(y_n,y_n,y_{n+1},t) *$
$M_1(x_{n-1},x_{n-1},x_n,t)$
$\geq M_2(y_n,y_n,y_{n+1},qt) * M_1(x_{n-1},x_{n-1},x_n,t)$
$\geq M_1(x_{n-1},x_{n-1},x_n,t)$

Thus we have

$M_2(y_n,y_n, y_{n+1},qt) \geq M_1(x_{n-1}, x_{n-1}, x_n,t)$ (4)

Therefore, by (3) and (4)

$M_1(x_n,x_n,x_{n+1},qt) \geq M_2(y_n,y_n,y_{n+1},t)$
$\geq M_1(x_{n-1}, x_{n-1},x_n,t)$
$\vdots$
$\geq M_1(x_0, x_0, x_1,t/q^n) \to 1$ as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M_1,*)$ is complete, $\{x_n\}$ converges to a point z in X. Similarly we prove $\{y_n\}$ converges to a point w in Y.

Again by (2) we have

$M_2(T_nz,T_nz,y_{n+1},qt) = M_2(T_nz,T_nz,T_nS_ny_n,qt)$
$\geq M_2(T_nz,T_nz,y_n,t) * M_2(y_n,y_n,T_nS_ny_n,t)$
$* M_1(z, z,S_ny_n,t)$
$= M_2(T_nz,T_nz,y_n,t) * M_2(y_n,y_n,y_{n+1},t) *$
$M_1(z, z, x_n,t)$ (5)

Letting $n \to \infty$ in (5) we have

$M_2(T_nz,T_nz,w,qt) \geq M_2(T_nz,T_nz,y_n,t) * 1 * 1$, That is $M_2(T_nz,T_nz,w,qt) \geq M_2(T_nz,T_nz,y_n,t)$ which implies that $M_2(T_nz,T_nz,w,qt) =1$ so that $T_nz = w$. On the other hand, by (1) we have

$M_1(S_nw, S_nw, x_{n+1},qt) = M_1(S_nw,S_nw,S_nT_nx_n,t)$
$\geq M_1(S_nw,S_nw,x_n,t) *$
$M_1(x_n,x_n,S_nT_nx_n,t) *$
$M_2(w,w, T_nx_n,t)$
$= M_1(S_nw,S_nw,x_n,t) * M_1(x_n,x_n,x_{n+1},t)$
$* M_2(w,w,y_{n+1},t)$ (6)

Letting $n\to\infty$ in (6) , it follows that $S_nw = z$. Therefore we have $S_nT_nz = S_nw = z$ and $T_nS_nw = T_nz = w$, which means that the point z is a fixed point of S_nT_n and the point w is a fixed point of T_nS_n. To prove the uniqueness of the fixed point z, let z' be the second fixed point of S_nT_n.

By (1) we have

$M_1(z ,z, z',qt) = M_1(S_nT_nz, S_n T_nz, S_nT_nz',qt)$
$= M_1(S_n (T_nz), S_n (T_nz), S_nT_nz',qt)$
$\geq M_1(S_nT_nz,S_nT_nz,z',t) *$
$M_1(z',z', S_nT_nz',t) * M_2(T_nz , T_nz , T_nz',t)$

$= M_1(z, z, z', t) * M_1(z', z', z', t) *$
$M_2(T_n z, T_n z, T_n z', t)$
$\geq M_2(T_n z, T_n z, T_n z', t)$

$M_2(T_n z, T_n z, T_n S_n T_n z', t) \geq M_2(T_n z, T_n z, T_n z', t) *$

$M_2(T_n z', T_n z', T_n S_n T_n z', t) * M_1(z, z, S_n T_n z', t)$

$= M_2(T_n z, T_n z, T_n z', t) *$

$M_2(T_n z', T_n z', T_n z', t) * M_1(z, z, z', t)$

$\geq M_2(T_n z, T_n z, T_n z', t) * 1 *$
$M_1(z, z, z', t)$
$\geq M_1(z, z, z', t)$

Hence $M_1(z, z, z', qt) \geq M_2(T_n z, T_n z, T_n z', t) \geq M_1(z, z, z', t)$ Thus $z = z'$. So the point z is the unique common fixed point of $\{S_n A_n\}$ and $\{T_n B_n\}$ in X. Similarly we prove w is a unique common fixed point of $\{B_n S_n\}$ and $\{A_n T_n\}$ in Y.

Corollary: 2.13 Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces with continuous t-norm* defined by $a*b = \min\{a,b\}$ for all $a, b \in [0,1]$. If T is a mapping from X into Y and S is a mapping from Y into X satisfying

$M_1(Sy, Sy, STx, qt) \geq M_1(Sy, Sy, x, t) * M_1(x, x, STx, t) *$

$M_2(y, y, Tx, t)$ $M_2(Tx, Tx, TSy, qt) \geq M_2(Tx, Tx, y, t) * M_2(y, y, TSy, t) *$
$M_1(x, x, Sy, t)$

for all x in X and y in Y where $q < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

Theorem 2.14: Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two complete M- fuzzy metric spaces with continuous t-norm* defined by $a*b = \min\{a,b\}$ for all $a, b \in [0,1]$. Let A_i, B_j be mappings of X into Y and S_p, T_q be mappings of Y into X satisfying the inequalities.

$$M_1(S_p A_i x, T_q B_j x', T_q B_j x', qt) \geq M_1(x, x', x', t) * M_1(x, x, S_p A_i x\ t) * M_1(x', x', T_q B_j x', t) * M_1(x, x, T_q B_j x', 2t) \quad (1)$$

$$M_2(B_j S_p y, A_i T_q y', A_i T_q y',,) \geq \{M_2(y.y', y', t) * M_2(y, y, B_j S_p y, t) * M_2(y', y', A_i T_q y', t), * M_2(y, y, A_i T_q y', 2t) \quad (2)$$

for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where $q<1$. If one of the mappings $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_n A_n\}$ and $\{T_n B_n\}$ have a unique common fixed point z in X and $\{B_n S_n\}$ and $\{A_n T_n\}$ have a unique common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w = \{T_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by
$A_n x_{2n-2} = y_{2n-1},\ S_n y_{2n-1} = x_{2n-1},\ B_n x_{2n-1} = y_{2n};\ T_n y_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .
Applying equality (1), we have

$M_1(x_{2n+1}, x_{2n}, x_{2n}, qt) = M_1(S_n A_n x_{2n}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt)$
$\geq M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) *$
$M_1(x_{2n}, x_{2n}, S_n A_n x_{2n}, t) *$
$M_1(x_{2n-1}, x_{2n-1}, T_n B_n x_{2n-1}, t) *$
$M_1(x_{2n}, x_{2n}, T_n B_n x_{2n-1}, \mathbf{2}t)$
$= M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) * M_1(x_{2n}, x_{2n}, x_{2n+1}, t) *$
$M_1(x_{2n-1}, x_{2n}, x_{2n}, t) * M_1(x_{2n}, x_{2n}, x_{2n}, \mathbf{2}t)$
$= M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) * M_1(x_{2n}, x_{2n+1}, x_{2n+1}, t)$

Which implies

$$M_1(x_{2n+1}, x_{2n}, x_{2n}, qt) \geq M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t) \quad (3)$$

Now

$M_1(x_{2n}, x_{2n-1}, x_{2n-1} qt) = M_1(x_{2n-1}, x_{2n}, x_{2n}, qt)$
$= M_1(S_n A_n x_{2n-2}, T_n B_n x_{2n-1}, T_n B_n x_{2n-1}, qt)$
$\geq M_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) *$
$M_1(x_{2n-2}, x_{2n-2}, S_n A_n x_{2n-2}, t) *$
$M_1(x_{2n-1}, x_{2n-1}, T_n B_n x_{2n-1}, t) *$
$M_1(x_{2n-2}, x_{2n-2}, T_n B_n x_{2n-1}, \mathbf{2}t)$

$= M_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) * M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)$
$* M_1(x_{2n-2},x_{2n-2}, x_{2n-1}, t)*M_1(x_{2n-1}, x_{2n}, x_{2n}, t)$
$= M_1(x_{2n-2}, x_{2n-1}, x_{2n-1}, t) * M_1(x_{2n-1}, x_{2n-1}, x_{2n},t)$

Which implies

$M_1(x_{2n},x_{2n-1},x_{2n-1},qt) \geq M_1(x_{n-1}, x_{n-2}, x_{n-2}, t/q)$ (4)

Thus from inequalities (3) and (4) we have

$M_1(x_{n+1},x_n, x_n,qt) \geq M_1(x_n,x_{n-1}, x_{n-1},t)$
$\geq M_1(x_{2n-1}, x_{2n-2}, x_{2n-2}, t/q)$
$\vdots$
$\geq M_1(x_1,x_0, x_0,t/q^{n-1}) \to 1$ as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequence in X. Since $(X, M_1,*)$ is complete, $\{x_n\}$ converges to a point z in X. Similarly applying inequality (2) and proceeding as above, we prove $\{y_n\}$ converges to a point w in Y. Suppose $\{A_n\}$ is continuous, then

$\lim_{n\to\infty} A_n x_{2n} = A_n z = \lim_{n\to\infty} y_{2n+1} = w$.

Now we prove $S_nA_nz = z$.

We have

$M_1(S_nA_nz, z, z ,qt) = \lim_{n\to\infty} M_1(S_nA_nz, T_nB_nx_{2n-1},T_nB_nx_{2n-1},qt)$
$\geq \lim_{n\to\infty} \{M_1(z,x_{2n-1}, x_{2n-1},t) *$
$M_1(z,z,S_nA_nz,t)*$
$M_1(x_{2n-1},x_{2n1},T_nB_nxss_{2n-1},t)*$
$M_1(z, z,T_nB_nx_{2n-1},2t) \}$
$= M_1(z, z, z, t) * M_1(z, S_nA_nz,t) *$
$M_1(z,z,z,t) * M_1(z, z,z,2t)$
$= 1 * M_1(z,S_nA_nz,t) * 1* 1$
$= M_1(z, z, S_nA_nz, t)$
$\geq M_1(z, z, S_nA_nz, t/q^n) \to 1$ as $n\to\infty$

Thus $S_nA_nz = z$. Hence $S_nw = z$. (Since $A_nz = w$) Now we prove $B_nS_nw = w$. We have

$M_2(B_nS_nw, w,w,qt) = \lim_{n\to\infty} M_2(B_nS_nw, y_{2n+1}, y_{2n+1},qt)$
$= \lim_{n\to\infty} M_2(B_nS_nw, A_nT_ny_{2n}, A_nT_ny_{2n}, qt)$
$\geq \lim_{n\to\infty} \{ M_2(w, y_{2n}, y_{2n},t) *$
$M_2(w,w,B_nS_nw,t)*$
$M_2(y_{2n},y_{2n},A_nT_ny_{2n},t)*$
$M_2(w, w, A_nT_ny_{2n},2t) \}$
$= M_2(w,w,w,t) * M_2(w, w, B_nS_nw, t) *$
$M_2(w,w,w,t) *M_2(w, w, w,2t)$
$\geq M_2(w, w, B_nS_nw,t/q^n) \to 1$ as $n\to\infty$

Thus $B_nS_nw = w$. Hence $B_nz = w$. (Since $S_nw = z$), Now we prove $T_nB_nz = z$.

$M_1(z, T_nB_nz,T_nB_nz,qt) = \lim_{n\to\infty} M_1(x_{2n+1}, T_nB_nz, T_nB_nz, qt)$
$= \lim_{n\to\infty} M_1(SAx_{2n}, T_nB_nz, T_nB_nz, qt)$
$\geq \lim_{n\to\infty} \{ M_1(x_{2n}, z, z,t) *$
$M_1(x_{2n},x_{2n},S_nA_nx_{2n},t)*$
$M_1(z,z,T_nB_nz,t)*$
$M_1(x_{2n}, x_{2n} , T_nB_nz,2t) \}$
$\geq M_1(z,z,T_nB_nz, t/q^n) \to 1$ as $n\to\infty$

Thus $T_nB_nz = z$. Hence $T_nw = z$. (Since $B_nz = w$), Now we prove $A_nT_nw = w$.

$M_2(w, A_nT_nw,qt) = \lim_{n\to\infty} M_2(y_{2n}, A_nT_nw, A_nT_nw,qt)$
$= \lim_{n\to\infty} M_2(B_nS_ny_{2n-1}, A_nT_nw, A_nT_nw,qt)$
$\geq \lim_{n\to\infty} \{M_2(y_{2n-1},w,w,t) *$
$M_2(y_{2n-1},y_{2n-1},B_nS_ny_{2n-1},t)*$
$M_2(w,w,A_nT_nw,t)*$
$M_2(y_{2n-1}, y_{2n-1}, A_nT_nw, 2t) \}$
$\geq M_2 (w, w, A_nT_nw,t/q^n) \rightarrow 1$ as $n\rightarrow\infty$

Thus $A_nT_nw = w$. The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

Uniqueness: Let z' be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X, w' be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

We have

$M_1(z, z', z',qt) = M_1(S_nA_nz, T_nB_nz', T_nB_nz',qt)$
$\geq M_1(z, z', z',t) * M_1(z, z, S_nA_nz,t) *$
$M_1(z', z', T_nB_nz',t) *M_1(z, z, T_nB_nz',2t)$
$= M_1(z, z',z',t) * M_1(z, z, z, t) *$
$M_1(z',z',z',t) * M_1(z, z, z',2t)$
$\geq M_1(z, z', z',t) * 1* 1 * M_1(z, z', z',t)$
$= M_1(z, z', z',t)$
$\geq M_1(z, z', z',t/ q^n) \rightarrow 1$ as $n\rightarrow\infty$

Thus $z = z'$. So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

Remark: 2.15 If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.8, we get the following corollary.

Corollary: 2.16 Let (X, M_1,*) and (Y, M_2,*) be two complete M- fuzzy metric spaces with continuous t-norm* defined by a*b = min{a,b}for all a, b $\in$[0,1]. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$M_1(SAx, TBx', TBx',qt) \geq M_1(x, x', x',t) * M_1(x, x, SAx\ t)$
$*M_1(x', x',TBx',t) * M_1(x, x,TBx',2t)$

$M_2(BSy, ATy', ATy',,qt) \geq M_2(y.y', y',t) * M_2(y, y, BSy ,t)$
$* M_2(y', y',ATy' ,t),* M_2(y, y, ATy',2t)$

for all x, x′ in X and y,y′ in Y where q<1. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Remark : 2.17 If (X, M_1,*) and (Y, M_2,*) are the same M- fuzzy metric spaces in the above theorem 2.8, then we obtain the following theorem as corollary.

Corollary: 2.18[10] Let (X, M,*) be a complete M- fuzzy metric space with continuous t-norm* defined by a*b = min{a,b} and $T_n : X \rightarrow X$ be a sequence of maps such that for all t > 0 and
0 < k < 1 satisfying the condition

$M (T_ix, T_jy, T_jy, t) \geq \{ M(x, y, y, t/k) * M(x, x, T_ix, t/k) *$
$M(y, y, T_jy, t/k) * M(x, x, T_jy, 2t/k)\}$

for all $i \neq j$ and for all x, y in X. Then $\{T_n\}$ have a unique common fixed point.

Theorem 2.19: Let (X, M_1,*) and (Y, M_2,*) be two complete M- fuzzy metric spaces with continuous t-norm* defined by a*b = min{a,b}for all a, b $\in$[0,1]. Let A_i, B_j be mappings of X into Y and S_p, T_q be mappings of Y into X satisfying the inequalities.

$M_1((S_pA_ix, T_qB_jx', T_qB_jx',qt) \geq \min\{ M_1(x,x', x',t),$
$M_1(x, S_pA_ix,S_pA_ix\ t), M_1(x', T_qB_jx',T_qB_jx',t),$
$M_2(A_ix,B_jx',B_jx',t),M_1(x, T_qB_jx',T_qB_jx',2t),$
$M_1(SAx,x', x',2t) \}$ (1)

$M_2(B_jS_py,A_iT_qy',A_iT_qy',,qt) \geq \min\{M_2(y.y', y',t),$
$M_2(y,B_jS_py,B_jS_py,t), M_2(y',A_iT_qy',A_iT_q\ y't),$

$M_1(S_py,T_qy',T_qy',t)$, $M_2(y,A_iT_qy',A_iT_qy',2t)$,
$M_2(B_jS_py,y',y',2t)$ } (2)
for all $i \neq j \neq p \neq q$, x, x′ in X and y,y′ in Y where q<1. If one of the mappings $\{A_n\}$,$\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ is continuous , then $\{S_nA_n\}$ and $\{T_nB_n\}$ have a unique common fixed point z in X and $\{B_nS_n\}$ and $\{A_nT_n\}$ have a unique common fixed point w in Y. Further, $\{A_n\}z = \{B_n\}z = w$ and $\{S_n\}w =$
$\{T_n\}w = z$.

Proof: Let x_0 be an arbitrary point in X and we define the sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by
$A_nx_{2n-2} = y_{2n-1}$, $S_ny_{2n-1} = x_{2n-1}$, $B_nx_{2n-1} = y_{2n}$; $T_ny_{2n} = x_{2n}$ for n = 1, 2, 3 . . . . .
Now we have

$$\begin{aligned}
M_1(x_{2n+1}, x_{2n}, x_{2n},qt) &= M_1(S_nA_nx_{2n}, T_nB_nx_{2n-1}, T_nB_nx_{2n-1},qt)\\
&\geq \min\{M_1(x_{2n}, x_{2n-1},x_{2n-1}, t),\\
&\quad M_1(x_{2n},S_nA_nx_{2n}, S_nA_nx_{2n},t),\\
&\quad M_1(x_{2n-1},T_nB_nx_{2n-1},T_nB_nx_{2n-1},t),\\
&\quad M_2(A_nx_{2n},B_nx_{2n-1}, B_nx_{2n-1},t),\\
&\quad M_1(x_{2n},T_nB_nx_{2n-1}, T_nB_nx_{2n-1}, \mathbf{2}t),\\
&\quad M_1(S_nA_nx_{2n},x_{2n-1}, x_{2n-1},2t) \}\\
&= \min\{ M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t),\\
&\quad M_1(x_{2n},x_{2n+1},x_{2n+1}, t),M_1(x_{2n-1},x_{2n},x_{2n},t),\\
&\quad M_2(y_{2n+1},y_{2n}, y_{2n},t), M_1(x_{2n}, x_{2n}, x_{2n}, \mathbf{2}t),\\
&\quad M_1(x_{2n+1}, x_{2n-1}, x_{2n-1}, \mathbf{2}t) \}\\
&\geq \min\{ M_1(x_{2n}, x_{2n-1}, x_{2n-1}, t),\\
&\quad M_1(x_{2n}, x_{2n+1}, x_{2n+1}, t), M_1(x_{2n-1}, x_{2n},t),\\
&\quad M_2(y_{2n+1},y_{2n},y_{2n},t),1,M_1(x_{2n+1},x_{2n},x_{2n},t)\\
&\quad *M_1(x_{2n}, x_{2n-1},x_{2n-1},t)\}\\
&\geq \min\{M_2(y_{2n+1},y_{2n},y_{2n},t),\\
&\quad M_1(x_{2n+1}, x_{2n},x_{2n},t)*M_1(x_{2n}, x_{2n-1},x_{2n-1},t) \}\\
&\geq \min\{ M_2(y_{2n+1},y_{2n},y_{2n},t),\\
&\quad M_1(x_{2n}, x_{2n-1},x_{2n-1},t) \}
\end{aligned}$$

Now

$$\begin{aligned}
M_2(y_{2n+1}, y_{2n},y_{2n},qt) &= M_2(B_nS_ny_{2n-1}, A_nT_ny_{2n}, A_nT_ny_{2n}, qt)\\
&\geq \min\{ M_2(y_{2n-1},y_{2n}, y_{2n},t),\\
&\quad M_2(y_{2n-1},B_nS_ny_{2n-1}, B_nS_ny_{2n-1},t),\\
&\quad M_2(y_{2n}, A_nT_ny_{2n}, A_nT_ny_{2n},t),\\
&\quad M_1(S_ny_{2n-1},T_ny_{2n},T_ny_{2n},t),\\
&\quad M_2(y_{2n-1}, A_nT_ny_{2n}, A_nT_ny_{2n}, 2t) ,\\
&\quad M_2(B_nS_ny_{2n-1},y_{2n}, y_{2n}, 2t)\}\\
&= \min\{ M_2(y_{2n-1},y_{2n},y_{2n},t),\\
&\quad M_2(y_{2n1},y_{2n},y_{2n},t),\\
&\quad M_2(y_{2n},y_{2n+1},y_{2n+1},t),\\
&\quad M_2(x_{2n-1}, x_{2n}, x_{2n},t),\\
&\quad M_2(y_{2n-1},y_{2n+1},y_{2n+1},2t),\\
&\quad M_2(y_{2n}, y_{2n}, y_{2n},2t)\}\\
&= \min\{ M_2(y_{2n-1}, y_{2n}, y_{2n},t),\\
&\quad M_2(y_{2n-1},y_{2n},y_{2n},t),\\
&\quad M_2 (y_{2n}, y_{2n+1},y_{2n+1},t),\\
&\quad M_1(x_{2n-1},x_{2n},x_{2n},t),\\
&\quad M_2(y_{2n-1},y_{2n},y_{2n},t)*\\
&\quad M_2 (y_{2n}, y_{2n+1},y_{2n+1},t), 1 \}\\
&\geq \min \{ M_2 (y_{2n-1},y_{2n},y_{2n},t),\\
&\quad M_1(x_{2n-1},x_{2n},x_{2n},t)\} \qquad (3)
\end{aligned}$$

Hencess
$M_1(x_{2n+1}, x_{2n}, x_{2n},qt) \geq \min \{ M_1(x_{2n-1}, x_{2n},x_{2n},t)$,

$M_2(y_{2n-1}, y_{2n},y_{2n},t)\}$ (4)

Similarly we have

$M_1(x_{2n}, x_{2n-1},x_{2n-1},qt) \geq \min\{ M_1(x_{2n-1}, x_{2n-2},x_{2n-2},t),$
$M_2(y_{2n-1}, y_{2n},y_{2n},t)\}$

$M_2(y_{2n},y_{2n-1},qt) \geq \min \{ M_2(y_{2n-1}, y_{2n-2},y_{2n-2},t),$
$M_1(x_{2n-1}, x_{2n-2}, x_{2n-2},t)\}$ (5)

Hence

$M_1(x_{2n}, x_{2n-1},x_{2n-1},qt) \geq \min \{M_1(x_{2n-1}, x_{2n-2}, x_{2n-2},t),$
$M_2(y_{2n-1}, y_{2n-2},y_{2n-2},t)\}$ (6)

from inequalities (3), (4), (5) and (6), we have

$M_1(x_{n+1},x_n, x_n,qt) \geq \min \{ M_1(x_n,x_{n-1}, x_{n-1},t),$
$M_1(y_n, y_{n-1}, y_{n-1},t/q)\}$
$\vdots$
$\geq\min\{M_1(x_1,x_0,x_0,t/q^{n-1}),M_2(y_1, y_0,y_0,t/q^n)\}$
$\to 1$ as $n\to\infty$

Thus $\{x_n\}$ is a Cauchy sequences in X. Since $(X, M_1,*)$ is complete, $\{x_n\}$ converges to a point z in X. Similarly we prove $\{y_n\}$ converges to a point w in Y. Suppose $\{A_n\}$ is continuous, then

$\lim_{n\to\infty} A_nx_{2n} = A_nz = \lim_{n\to\infty} y_{2n+1} = w$. Now we prove $S_nA_nz = z$.

Suppose $S_nA_nz \neq z$. We have

$M_1(S_nA_nz,z, z,qt) = \lim_{n\to\infty} M_1(S_nA_nz, T_nB_nx_{2n-1},T_nB_nx_{2n-1},qt)$
$\geq \lim_{n\to\infty} \min\{M_1(z,x_{2n-1}, x_{2n-1},t),$
$M_1(z,S_nA_nz, S_nA_nz,t), M_1(x_{2n-1},T_nB_nx_{2n-1},T_nB_nx_{2n-1},t),$
$M_2(A_nz,B_nx_{2n-1},B_nx_{2n-1},t),M_1(z,T_nB_nx_{2n-},T_nB_nx_{2n-1},2t),$
$M_1(S_nA_nz, x_{2n-1}, x_{2n-1},2t) \}$
$= \min\{M_1(z,z,z,t),M_1(z, S_nA_nz, S_nA_nz,t),$
$M_1(z,z,z,t), M_2(w,w,w,t),$
$M_1(z,z,z,2t), M_1(S_nA_nz, z, z,2t)\}$
$= \min\{ 1,M_1(z,S_nA_nz,t), 1, 1, 1,$
$M_1(S_nA_nz, z, z, 2t) \}$
$> M_1(z, S_nA_nz, S_nA_nz, t)$ (since $q < 1$) which is a contradiction.

Thus $S_nA_nz = z$. Hence $S_nw = z$. (Since $A_nz = w$) Now we prove $B_nS_nw = w$.

Suppose $B_nS_nw \neq w$. We have

$M_2(B_nS_nw, w,w,qt) = \lim_{n\to\infty} M_2(B_nS_nw, y_{2n+1}, y_{2n+1},qt)$
$= \lim_{n\to\infty} M_2(B_nS_nw, A_nT_ny_{2n}, A_nT_ny_{2n}, qt)$
$\geq \lim_{n\to\infty} \min\{M_2(w, y_{2n}, y_{2n},t),$
$M_2(w,B_nS_nw,B_nS_nw,t),M_2(y_{2n},A_nT_ny_{2n},A_nT_ny_{2n},t),$
$M_1(S_nw,T_ny_{2n},T_ny_{2n},t),M_2(w,A_nT_ny_{2n}, A_nT_ny_{2n},2t),$
$M_2(B_nS_nw, y_{2n}, y_{2n},2t) \}$
$= \min\{M_2(w,w,w,t), M_2(w,B_nS_nw, B_nS_nw,t),$
$M_2(w,w,w,t), M_1(z, z, z,t),$
$M_2(w, w, w,2t), M_2(B_nS_nw, w,w, 2t) \}$
$> M_2(w,B_nS_nw,B_nS_nw,t)$ (Since $q < 1$) which is a contradiction.

Thus $B_nS_nw = w$. Hence $B_nz = w$. (Since $S_nw = z$) Now we prove $T_nB_nz = z$.

Suppose $T_nB_nz \neq z$.

$M_1(z, T_nB_nz,T_nB_nz,qt) = \lim_{n\to\infty} M_1(x_{2n+1}, T_nB_nz,T_nB_nz,qt)$

$$= \lim_{n\to\infty} M_1(SAx_{2n}, T_nB_nz, T_nB_nz, qt)$$
$$\geq \lim_{n\to\infty} \min\{ M_1(x_{2n}, z, z, t), M_1(x_{2n}, S_nA_nx_{2n}, S_nA_nx_{2n}, t), M_1(z, T_nB_nz, T_nB_nz, t), M_2(A_nx_{2n}, B_nz, B_nz, t), M_1(x_{2n}, T_nB_nz, T_nB_nz, 2t), M_1(z, S_nA_nx_{2n}, S_nA_nx_{2n}, 2t)\}$$
$$= \min\{M_1(z,z,z,t), M_1(z,z,z,t), M_1(z,T_nB_nz,T_nB_nz,t), M_2(w,w,B_nz,t), M_1(z,T_nB_nz,T_nB_nz,2t), M_1(z, z, z,2t)\}$$
$$= \min\{ 1, 1, M_1(z,T_nB_nz, T_nB_nz,t), 1, M_1(z,T_nB_nz, T_nB_nz,2t), 1 \}$$
$$> M_1(z,T_nB_nz, T_nB_nz,t) \text{ (Since } q< 1)$$

which is a contradiction. Thus $T_nB_nz = z$. Hence $T_nw = z$. (Since $B_nz = w$)

Now we prove $A_nT_nw = w$ Suppose $A_nT_nw \neq w$.

$$M_2(w, A_nT_nw,qt) = \lim_{n\to\infty} M_2(y_{2n}, A_nT_nw, A_nT_nw,qt)$$
$$= \lim_{n\to\infty} M_2(B_nS_ny_{2n-1}, A_nT_nw, A_nT_nw,qt)$$
$$\geq \lim_{n\to\infty} \min\{M_2(y_{2n-1},w,w,t), M_2(y_{2n-1},B_nS_ny_{2n-1},B_nS_ny_{2n-1},t), M_2(w,A_nT_nw,A_nT_nw,t), M_1(Sy_{2n-1},T_nw,T_nw,t), M_2(y_{2n-1},A_nT_nw,A_nT_nw,2t), M_2(B_nS_ny_{2n-1}, B_nS_ny_{2n-1},w, 2t) \}$$
$$\geq \min\{ M_2(w, w,w,t), M_2(w, w,w,t), M_2(w, A_nT_nw, A_nT_nw,t), M_1(z,z,z,t), M_2(w,A_nT_nw, A_nT_nw,t), M_2(w, w,w,2t) \}$$
$$> M_2 (w,A_nT_nw, A_nT_nw,t) \text{ (Since } q < 1)$$ which is a contradiction.

Thus $A_nT_nw = w$.

The same results hold if one of the mappings $\{B_n\}$,$\{S_n\}$ and $\{T_n\}$ is continuous.

Uniqueness: Let z' be another common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X, w' be another common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

We have

$$M_1(z, z', z',qt) = M_1(S_nA_nz, T_nB_nz', T_nB_nz',qt)$$
$$\geq \min\{M_1(z,z',z',t), M_1(z,S_nA_nz,S_nA_nz,t), M_1(z',T_nB_nz',T_nB_nz',t), M_2(A_nz,B_nz',z',t), M_1(z,T_nB_nz',T_nB_nz',2t), M_1(S_nA_nz ,z', z',2t)\}$$
$$\geq \min\{M_1(z, z' ,z ',t), M_1(z, z, z, t), M_1(z',z', z',t), M_2(w,w', w',t), M_1(z, z', z',2t), M_1(z, z', z',2t)\}$$
$$= \min\{ M_1(z, z', z', t), M_1(z, z, z, t), M_1(z',z', z',t), M_2(w,w',t), M_1(z, z', z',2t) \}$$
$$= \min\{M_1(z,z',z',t), 1, 1, M_2(w,w', w',t), M_1(z, z', z',2t)\}$$
$$> M_2(w,w', w',t) \text{ (Since } q < 1)$$
$$M_2(w, w', w',qt) = M_2(B_nS_nw,A_nT_nw', A_nT_nw',qt)$$
$$\geq \min\{M_2(w,w' w',t), M_2(w,B_nS_nw,B_nS_nw,t), M_2(w',A_nT_nw',A_nT_nw',t), M_1(Sw,Tw',Tw',t), M_2(w,A_nT_nw',A_nT_nw',2t), M_2(B_nS_nw,w',2t) \}$$
$$= \min\{ M_2(w,w', w't), M_2(w,w,w,t),$$

$M_2(w',w',w't), M_1(z,z',z',t),$
$M_2(w', w',2t), M_2(w', w',2t)\}$
$= \min\{ M_2(w,w', w',t), 1, 1,$
$M_1(z, z', z',t), 1, 1\}$
$> M_2(w,w', w't)$

Hence $M_1(z, z', z',qt) > M_2(w,w', w't) > M_2(w,w', w't)$
Which is a contradiction. Thus $z = z'$.
So the point z is the unique common fixed point of $\{S_nA_n\}$ and $\{T_nB_n\}$ in X. Similarly we prove w is a unique common fixed point of $\{B_nS_n\}$ and $\{A_nT_n\}$ in Y.

Remark : 2.20 If we put $A_i = A$, $B_j = B$, $S_p = S$ and $T_q = T$ in the above theorem 2.13, we get the following corollary.

Corollary: 2.21 Let (X, M_1,*) and (Y, M_2,*) be two complete M- fuzzy metric spaces with continuous t-norm* defined by $a*b = \min\{a,b\}$for all a, b $\in [0,1]$. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities.

$M_1((S_pA_ix, T_qB_jx', T_qB_jx',qt) \geq \min\{ M_1(x,x', x',t),$
$M_1(x, S_pA_ix,S_pA_ix\ t), M_1(x', T_qB_jx',T_qB_jx',t),$
$M_2(A_ix,B_jx',B_jx',t),M_1(x,T_qB_jx',T_qB_jx',2t),$
$M_1(SAx,x', x',2t) \}$

$M_2(B_jS_py, A_iT_qy', A_iT_qy',,qt) \geq \min\{M_2(y.y', y',t),$
$M_2(y, B_jS_py, B_jS_py ,t), M_2(y',A_iT_qy',A_iT_q\ y't),$
$M_1(S_py,T_qy',T_qy',t),M_2(y,A_iT_qy',A_iT_qy',2t),$
$M_2(B_jS_py,y', y',2t) \}$

for all x, x′ in X and y,y′ in Y where q<1. If one of the mappings A, B, S and T is continuous , then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, Az = Bz = w and Sw = Tw = z.

Remark : 2.22 If (X, M_1,*) and (Y, M_2,*) are the same M- fuzzy metric spaces in the above theorem 2.19, then we obtain the following theorem as corollary.

Corollary: 2.23[7] Let S and T be two self mappings of a complete *M*-fuzzy metric space (X, *M*, *). If there exists a number k ϵ (0, 1) such that
$M(Sx, Ty, Ty, kt) \geq \min\{M(x, y, y, t), M(x, Sx, Sx, t),$
$M(y, Ty, Ty, t), M(x, x, Sx, t), M(y, y, Ty, t)\}$
for all x, y ϵ X and t > 0, then S and T have a unique common fixed point..

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