Research Article

Renormalization of one dimensional discrete nonlinear map

Tarini Kr Dutta[1], Jayanta Kr Das[2*], Mridul Chandra Das[3]

[1]Professor, [3]Research Scholar, Department of Mathematics, Gauhati University, Guwahati, Assam, INDIA.
[2]Assistant Professor, Department of Mathematics, K.C. Das Commerce College, Guwahati, Assam, INDIA.
Email: dipali.das371@gmail.com

Abstract

In this paper we highlight the Feigenbaum renormalization theory through the non linear discrete map $f(x) = \mu\, x\,(1-x^4)$ and determine the scaling ratio (the Feigenbaum number α) as the period doubling sequence proceeds towards chaos. Computer software packages 'Mathematica' and 'C-programming' are used prudentially to implement numerical algorithms for our purpose.

Keywords*:* Periodic points, Bifurcation points, renormalization, Feigenbaum number.

***Address for Correspondence:**
Dr. Jayanta Kr Das, Assistant Professor, Department of Mathematics, K.C. Das Commerce College, Guwahati, Assam, INDIA.
Email: dipali.das371@gmail.com

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INTRODUCTION[11,14,16]

A map may be considered as a discrete dynamical system and its next iteration depends on previous iterations. Maps have similar characteristics to differential equations, such as stable and unstable fixed points, limit cycles, bifurcations via Poincare's section. Period doubling is a process in which certain maps are subjected to it. A few common underlying numerical characteristics or scaling laws have been initiated in all period-doubling sequences whether observed in one or higher dimensional maps, fluids or in electromagnetism or the like. He was Mitchell Feigenbaum who made this remarkable observation of universality related to period-doubling. Feigenbaum used the powerful mathematical technique of renormalization group (RG) method suited for understanding transitions with scaling properties. In the paper "Universal Behavior in Nonlinear Systems", he calculates two universal constants or indices α and δ, called the Feigenbaum indices. In period doubling maps there exists a value of the parameter for which there are fixed points (one stable), but no cycle. As the parameter is increased, a 2-cycle appears from the fixed point, and the fixed point changes from stable to unstable. This is called flip bifurcation. As the parameter is further increased, another flip bifurcation occurs. Here a 4-cycle appears.

History[4,10]

Grossmann and Thomae (1977) and Feigenbaum (1978) discovered Feigenbaum scaling independently and almost simultaneously. In spite of being a purely mathematical phenomenon, it was discovered by theoretical physicists. Though they were mathematicians, they contributed to the subsequent development of the theory in physics .It is noticed that

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most of the resulting papers have appeared in the physics literature. The discovery of Feigenbaum scaling aroused much excitement in the late 1970s, nearly for ten years. Parallely, it helped in the development of nonlinear dynamical systems theory. By the late 1980s most physicists appeared to have decided that the subject was completely understood, and little more remained to be done. Feigenbaum scaling has generic influence in all systems except those in which it does not occur.

Definition[4]: Let $\{f_\mu\}$be the family of iterated maps on a space X. Then discrete dynamical system defined as $f_\mu : x \rightarrowtail f_\mu(x) \ldots\ldots(1)$,where X may be real R, the planeR^2, the complex numbers$C$,the quaternion$H$,the circle T^1or two-torusT^2 and μ taken as control parameter. μ is the real ,complex, or quaternion number and varied slowly compared to the rate of iteration of f_μ in experiment modeled by equation (1).

Definition [4]: The sequence $\{x_1, x_2, x_3, x_4, \ldots\ldots\ldots\ldots\}$ is called the orbit of the seed x_0 where$x_n = f_\mu(x_{n-1}), n = 1,2,3, \ldots\ldots\ldots$....and nth iterate of x_n denote here$f^{\langle n \rangle}(x_0)$.

Definition[4]: An $n - cycle$ is an orbit with $x_n = x_0$ for some integer $n > 0$(the period).

Theorem[4]: Every $n - cycle$ has a stability(ρ) in case of $X = \mathcal{R}\ or\ \mathbb{C}$ where $\rho = \prod_{i=0}^{n-1} f_\mu^{/}(x_0)$. The orbits are super stable if$\rho = 0,$ and stable if $|\rho| < 0.$

Feigenbaum scaling[2,4,7,11,12]

The theory of Feigenbaum scaling is concerned with the behavior of super stable orbits, in particular, with rate of variation of μ as the period $n \to \infty$. To make sense in this purpose, we need to study the family of maps of the following properties…….

1. The maps possess a complete sequence of period doubling bifurcations.
2. We may find an infinite sequence of parameters values $\{\mu_1, \mu_2, \mu_3, \ldots\ldots\ldots\}$such that a stable cycle of period 2^k of f_μ bifurcates to a stable2^{k+1}cycle at $\mu = \mu_k$.

In 1978 Feigenbaum made the original discovery that the limit $\delta = \lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$ exist by taking the standard example on the real line is the logistic family $f_\mu = \mu\, x\,(1 - x)$ .For this limit, he computed an approximate value 4.6692 and also showed that there exists an orbit scaling$(\alpha) = \lim_{n\to\infty} \frac{d_n}{d_{n+1}}$, where d_n is the value of the nearst cycle to zero in the 2^k cycle. The scaling value(α) determined by Feigenbaum is about -2.5029…...The constants δ, α are familiarly known as '$Feigen\,Values$'.

RENORMALIZATION THEORY[4,19]

Renormalization is any of a collection of techniques used to treat infinities arising in calculated quantities. The renormalization theory is based on the self-similarity of the $figtree$. Feigenbaum in his research papers explain the scaling behavior in terms of the period doubling operator as follows: Consider the period doubling operator T and defined as $f(x) \to (Tf)(x) \equiv \alpha f \circ f(\alpha^{-1}x) \to (3)$ where $\alpha^{-1} \equiv f(0)$. Then Feigenbaum proposed scaling behavior in terms of a function fixed by T so that $\alpha g \circ g(\alpha^{-1}x) = g(x) \to (4)$ .The equation (4)is known as $Feigenbaum's\ Functional\ Equation(FFE)$ and the operator T is known as $period - doubling\ renormalization\ operator$..Predrag Cvitanovic had a significant contribution in the early stages of this work. His contribution was that for large integers k, appropritely scaled versons of the functions$f_{\mu_k}^{2^{k-1}}$and $f_{\mu_{k+1}}^{2^k}$ should approach each other, at least near zero, for any suitable family f_μ ; and that the limiting function $g(x) = \lim_{k\to\infty} g_k(x)$ where$g_k(x) = (T^k f)(x) = \alpha^k f_{\mu_k}^{2^k}(\alpha^{-k}x)$ should be independent of the familyf_μ and should satisfy the functional equation(4).Therefore, $Feigenbaum's\ Functional\ Equation(FFE)$ is also known as Cvitanovic-Feigenbaum equation. In a given dynamical system, all scaling properties of period doubling can be deduced from the appropriate solution g of the FFE ,therefore each solution g is called a $universal\ function$. The linearization DT_g of T is important in this work. The expression $DT_g h)(x) = \alpha g^{/} \circ g(\alpha^{-1}x)h(\alpha^{-1}x) + \alpha h \circ g(\alpha^{-1}x)\{(\alpha^2 - 1)h(0) + \alpha h(1)\}\{xg^{/}(x) - g(x)\}$ is the linearization DT_g of T about the fixed point g whose action on a function h.

Renormalization Procedure[15,19]

Consider $f(x, \mu)$ denote a discrete unimodal map that undergoes a period- doubling route to chaos as μ increases. Let x_k is the maximum of f and μ_n denote the values of μ at which a 2^n cycle is born. Again, let R_n denote the values μ at which the2^n cycle is super stable. Also, let d_n denote the distance between the x values where maximum occurs and the closest fixed point to this x value that lays on the cycle ($i.e\ d_n = f_{R_n}^{2^{n-1}}(x_k) - x_k$) and d_n decreases with each period doubling. Then $\frac{d_n}{d_{n+1}} \to a\ universal\ limit$ as $n \to \infty$ i.e. $\frac{d_n}{d_{n+1}} \to \alpha$. The value of α for the logistic map is (-2.502907875). Here the negative sign indicates that the nearest point in the 2^n cycle is alternatively above and below. We know the renormalization process depends on the self-similarity of the $figtree$ then this structure indicates the endless repetition of the following dynamical processes

1. a 2^n-cycle is born,
2. then becomes super stable,
3. and then losses stability in a period- doubling bifurcation

Qualitative observation

To express the self similarity mathematically, we compare f with its second iterate f^2 at corresponding values of μ, then renormalize one map into the other. By comparison of these two maps we get the following properties

1. x_k is a super stable fixed point for both of them.
2. μ increased from R_0 to R_1 in the second iterate of f, which is the basic part of renormalization procedure.
3. the maps (i.e. $f(x, R_0), f^2(x, R_1)$) cobweb diagrams starting from corresponding points would look the almost same.

Qualitative Observations Conversion into Formulas

Redefining x as $x - x_k$ to translate the origin of x to x_k.This implies that we can subtract x_k from , since $f(x_n, r) = x_{n+1}$ and then d_n can be written as $D_n = f_{R_n}^{(2^{n-1})}(0)$.

To make figure $f^2(x, R_1)$ look like figure$f(x, R_0)$, we blow it up by a factor $|\alpha| < 1$ in both directions and in place of (x, y) putting $(-x, -y)$ can invert it. Lastly, the likeness between the figures of $f(x, R_0)\ and\ \alpha f^2(\frac{x}{\alpha}, R_1)$ shows that $f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$ (5)

From this we conclude that by taking second iterate of$f(i.e. f^2)$, f has been renormalized, rescaling $x \to \frac{x}{\alpha}$,and shifting μ to the next super stable value. Similarly, we can renormalized f^2 to generate f^4 and also it has a super stable fixed point if we shift μ to R_2.By the same reasoning we can claim that

$$f^2\left(\frac{x}{\alpha}, R_1\right) \approx \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right) \quad (6)$$

In terms of the original map $f(x, R_0)$, the equation (6) can be expressed as $f(x, R_0) \approx \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$ ... ... (7)

Thus, renormalizing after n times we get

$$f(x, R_0) \approx \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) \ldots \ldots \ldots \ldots (8)$$

In this case, Feigenbaum numerically proposed that $\lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x)$, which is a universal function.

PROPOSED MAP AND RENORMALIZATION PROCEDURE[15,16, 19]

Determination of Feigenbaum $'\alpha'$:

Here we consider the non linear discrete model of the form $f(x) = \mu\, x\,(1-x^4)$ (9)
with control parameter $\mu\varepsilon[0,3]and\ x\varepsilon[-1,1]$.

Let us consider a super stable cycle of period 2^n in (9) and since$f^{/}\left(\frac{1}{5^{\frac{1}{4}}}\right) = 0$, the cycle contain the critical point $x_k = \frac{1}{5^{\frac{1}{4}}} = 0.66874$.

$\therefore$ The element nearest to x_k=the (2^{n-1})th iterate of x_k

So, $d_n = f_{R_n}^{2^{n-1}}\left(\frac{1}{5^{\frac{1}{4}}}\right) - \left(\frac{1}{5^{\frac{1}{4}}}\right)$, where n=1,2,3……

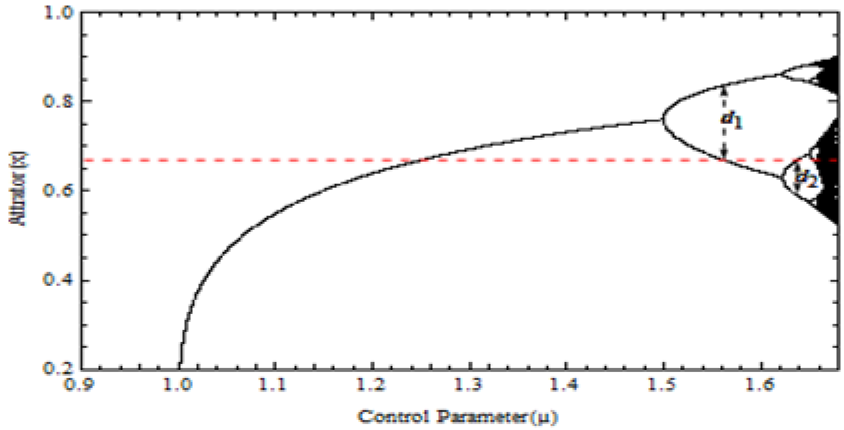

Figure 1: *Bifurcation diagram of the propsed map and d_n is the distance from x_k to the nearest point in 2^n cycle*

Now, for the equation (9) ,consider a $2-cycle$ is $\{x_1, x_2\}$. Then we get $x_1 = \mu\, x_2(1-x_2^4)$, $x_2 = \mu\, x_1(1-x_1^4)$
$\therefore\ x_1x_2 = \mu^2\, x_1\, x_2(1-x_1^4)(1-x_2^4)$
$\Rightarrow 1 = \mu^2(1-x_1^4)(1-x_2^4)$ (10)
If x_1 is supper attracting point of $2-cycle$,then we get
$(f^2)^/(x_1) = f^/(x_1)f^/(x_2) = 0$
$\Rightarrow\ \mu^2(1-5x_1^4)(1-5x_2^4) = 0$
$\Rightarrow\ either\ x_1 = \frac{1}{5^{\frac{1}{4}}}$ or $x_2 = \frac{1}{5^{\frac{1}{4}}}$
Let $x_1 = \frac{1}{5^{\frac{1}{4}}}$ follows then $x_2 = f(x_1) = \mu\, x_1(1-x_1^4) = \frac{4\mu}{5^{\frac{5}{4}}}$
Putting $x_2 = \frac{4\mu}{5^{\frac{5}{4}}}$ in the equation (10), we get
1-$\frac{4\,\mu^2}{5} + \frac{1024\,\mu^6}{15625} = 0 \ldots\ldots\ldots\ldots(11)$
When $\mu = \frac{5}{4} = 1.25$, then in the equation (9), the function f has super attracting point and therefore $\mu - \frac{5}{4}$ is a factor of the equation (11).
$\therefore\ (11) \Rightarrow (\ \mu - 1.25\ )(\mu + 1.25)(\mu + 1.562026728)\ (\mu - 1.562026728\ )\ (\mu + i2.0006)(\mu - i2.0006) = 0$ ..(12)
$\therefore\ (12) \Rightarrow \mu = 1.562026728$, which claims that the super attracting point for period $2-cycle$ occurs at $\mu = 1.562026728$.
Now, with the help of c-programming and using same above technique, we get the other super attracting fixed points for periods 4,8,16,32 … … . *etc* are found as follows

Periods	Super stable parameter values (R_n)	$f_{R_n}^{2^{n-1}}\left(\frac{1}{5^{\frac{1}{4}}}\right)$	$d_n = f_{R_n}^{2^{n-1}}\left(\frac{1}{5^{\frac{1}{4}}}\right) - \left(\frac{1}{5^{\frac{1}{4}}}\right)$	$\frac{d_n}{d_{n+1}}$
2	1.562026728	0.66874629	0.00000629	
4	1.637863254	0.668743026	− 0.000003026	−2.0786516853
8	1.654485566	0.668741436	0.000001436	− 2.1072423398
16	1.658101969	0.6687393375	−0.00000066250	−2.167543612
32	1.658877578	0.6687402912	0.000000291291	−2.2743579440
64	1.659043735	0.66873987825	−0.000000121753	−2.39247492874
128	1.659079323	0.66874004898	0.0000000489805	−2.485769701191
256	1.6590869455	0.668739980413	−0.0000001959165	−2.50028755487

Qualitative Observations

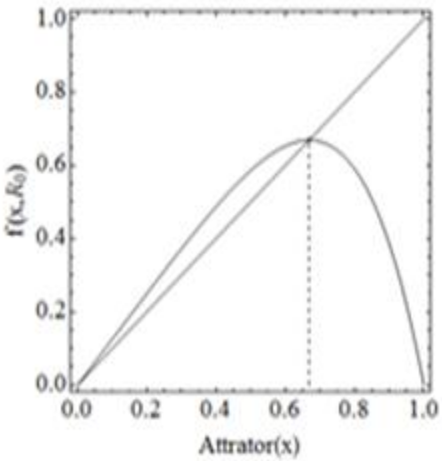

Figure 2(a): $x_k = 0.66874$ is the super stable fixed point for $f(x, R_0)$ where $R_0 = 1.25$

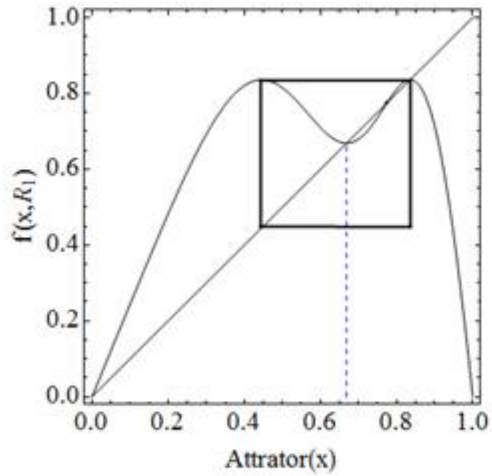

Figure 2(b): $x_k = 0.66874$ is the super stable fixed point for $f(x, R_1)$ where $R_1 = 1.562026728$.

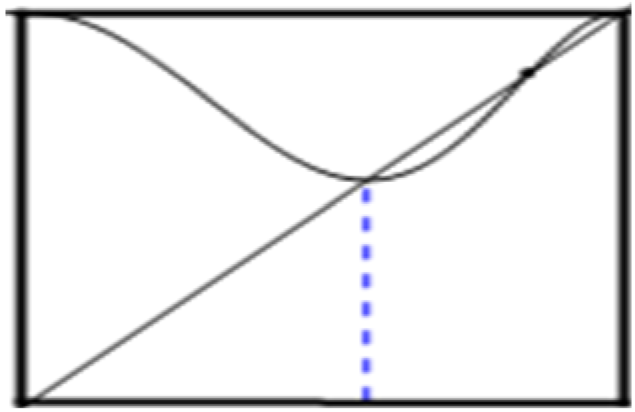

Figure 2(c): the small box portion of $f(x, R_1)$

From the observation of figures, we conclude that the look of $Figure\ 2(c)$ practically identical to $Figure\ 2(a)$. This implies that the self-similarity exist in between $f(x, R_0)\ and\ f(x, R_1)$.

Qualitative Observations Conversion into Formulas

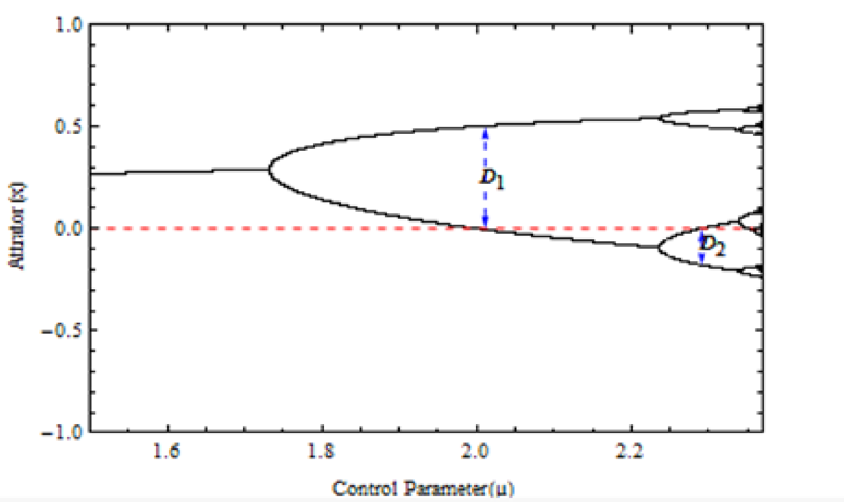

Figure2(d)：Bifurcation diagram of the propsed map when the super stable point is at origin and D_n is the distance from origion to the nearest point in2^ncycle

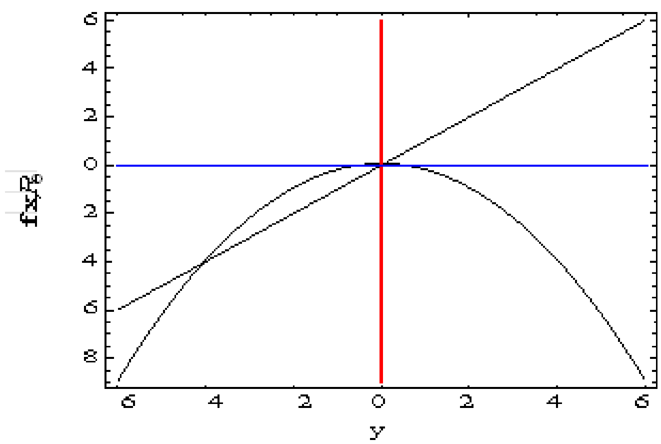

Figure 2(d): the diagram off(x, R_0) of the propsed map when the super stable point is at origin

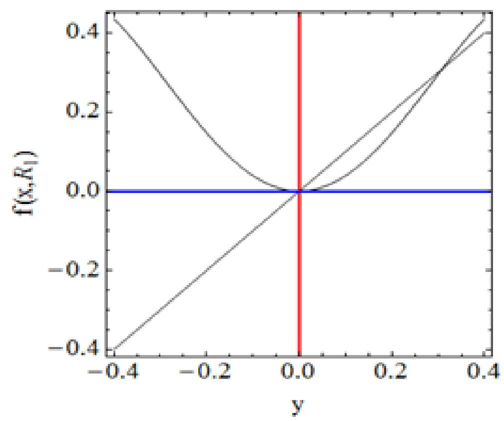

Figure2 (e): the portion of $f^2(x, R_1)$ofthe propsed map when the super stable point is at the origin

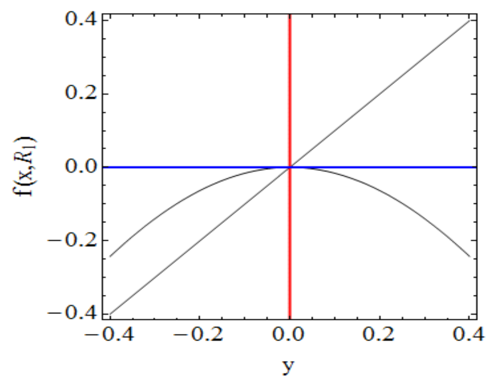

Figure2(f): the diagram of $\alpha f^2\left(\frac{x}{\alpha}, R_1\right)$ ofthe propsed map

From the resemblance between the figure $2(d), 2(f)$ shows that $f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$ and similarly we can prove others also.

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