Original Article

On Strength Reliability for Erlang-Truncated Exponential Distributed Stress

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Abstract Stress-strength reliability models are considered to estimate the reliability of the manufacturing devices facing Erlang-truncated exponential distribution. The strength of the manufacturing items follows power distribution and the stress follows Erlang-truncated exponential distribution. A stress- strength relationship among the parameters is established and further, these results are used to find the optimum cost when the cost function is linear in terms of parameters.

Key Words: Erlang- truncated exponential distribution, power function distribution, stress-strength reliability, incomplete gamma function and linear cost function.

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Received Date: 12/10/2017 Revised Date: 19/11/2017 Accepted Date: 04/12/2017

Access this article online	
Quick Response Code:	Website: www.statperson.com
	Accessed Date: 18 January 2018

INTRODUCTION

Reliability is defined as the probability that a system, component or device will performedequately itsintended function under the operational conditions. Improving reliability is an interesting topic of connotationfor the researches. Reliability function $R(t)$ which is a function of time 't' represents the probability that a device or item is still working at time 't'and is defined as$R(t) = P(X > Y) = 1 - F(t)$. One of the statistical tool to measure the reliability is the stress-strength reliability model, which is denoted as $P = Pr(X > Y)$. The stress-strength reliability model measure the performance of an item of strength Y subject to a stress X, where X and Y are taken to be non-negative continuous random variables. The term stress-strength was first introduced by Church and Harris (1970)[4]. For more references on the topic, one may referred to Downton (1973)[7], Tong (1974)[10], Kelly (1976)[8], Sathe and Shah (1981)[9], Chao (1982)[5], Awad (1986)[2], Chaturvedi and Surinder (1999)[6].El-Alosey (2007)[3] proposed Erlang-truncatedexponential distribution, denoted by $ETE(\beta,\lambda)$. A random variable (r.v.) X is said to follow ETE distribution if its probability density function (pdf) is given by

$$f(x;\beta,\lambda) = \beta\left(1-e^{-\lambda}\right)e^{-\beta x\left(1-e^{-\lambda}\right)}, x \geq 0, \beta > 0, \lambda > 0, \quad (1.1)$$

where $\beta$and λ are the shape and scale parameters respectively. In the present study, Erlang-truncatedexponential distribution has been considered to study the strength-reliability of an item for Erlang-truncatedexponential distributed stress.

STRENGTH RELIABILITY FOR FINITE STRENGTH

A finite random time power function distribution 'Y' distribution has been chosen to represent the strength which justify the fact that the life time of the items or devices may confined to a finite range only. On the other hand an infinite range of stress 'X' may be justifiable due to the fact that large stress may tend towards infinity. Here, let 'Y' be a random variable which denotes the strength of an item that follows power function distribution with pdf

$$g(y;a,\theta) = \left(\frac{a}{\theta}\right)\left(\frac{y}{\theta}\right)^{a-1}, 0 < y < \theta, a > 0 \quad (2.1)$$

How to site this article: Surinder Kumar, Ajay Kumar. On Strength Reliability for Erlang-Truncated Exponential Distributed Stress. *International Journal of Statistika and Mathemtika*. November to January 2018; 25(1): 19-23. http://www.statperson.com

where 'θ' is the scale and 'a' is shape parameter respectively. The maximum value of strength is θ. Thus, the total unreliability of the devices/items is obtained by P(X>Y), where $\theta > Y$, which is termed as probability of disaster by Alam and Roohi (2003)[1]. In the above setup, underlying problem may handle mathematically as follows

$$\alpha = P(X > \theta) = \int_{\theta}^{\infty} f(x;\beta,\lambda)dx$$

$$= \int_{\theta}^{\infty} \beta(1-e^{-\lambda})e^{-\beta x(1-e^{-\lambda})}\,dx$$

on substituting $\beta x(1-e^{-\lambda}) = t$, we get

$$\alpha = P(X > \theta) = e^{-m(1-e^{-\lambda})} \quad (2.2)$$

where $m = \beta\theta$

Table 1: Numerical values for Probability of disaster $\propto = P(X > \theta)$ for different values of m and λ

m	$P(X>\theta)$ $\lambda=0.5$	$\lambda=1$	$\lambda=1.5$	$\lambda=2$	$\lambda=2.5$	$\lambda=3$	$\lambda=3.5$
0.1	0.961417	0.938744	0.925254	0.917166	0.912295	0.909354	0.907574
0.2	0.924323	0.881241	0.856095	0.841194	0.832283	0.826924	0.82369
0.3	0.88866	0.82726	0.792105	0.771515	0.759288	0.751966	0.74756
0.4	0.854373	0.776586	0.732899	0.707607	0.692695	0.683803	0.678466
1	0.674712	0.531464	0.459843	0.421193	0.399351	0.386659	0.379158
1.5	0.554214	0.387445	0.311828	0.273351	0.252367	0.240432	0.233469
2	0.455236	0.282454	0.211456	0.177403	0.159481	0.149505	0.143761
2.5	0.373935	0.205913	0.143392	0.115134	0.100783	0.092965	0.088522
3	0.307153	0.150114	0.097236	0.074721	0.063689	0.057807	0.054508
3.5	0.252298	0.109435	0.065938	0.048493	0.040248	0.035946	0.033564
4	0.20724	0.07978	0.044714	0.031472	0.025434	0.022352	0.020667
4.5	0.170229	0.058161	0.030321	0.020425	0.016073	0.013899	0.012726
5	0.139827	0.0424	0.020561	0.013256	0.010157	0.008642	0.007836
5.5	0.114855	0.03091	0.013943	0.008603	0.006419	0.005374	0.004825
6	0.094343	0.022534	0.009455	0.005583	0.004056	0.003342	0.002971
6.6	0.074504	0.015421	0.005932	0.003323	0.002339	0.00189	0.00166
7	0.063654	0.011976	0.004348	0.002352	0.00162	0.001292	0.001127
7.7	0.048329	0.007694	0.002524	0.001284	0.000852	0.000664	0.000571

Table 2: Values of m for tolerance levels α and for fixed value of $\lambda = 0.5$

$\boldsymbol{\alpha}$	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
$\boldsymbol{m}$	2.541494	3.30656	4.317922	5.082988	7.624482	10.16598	12.70747

Remarks

1. Table 1 depicts the probability of disaster for Erlang-truncated exponential distributed stress. It is interested to note that the probability of disaster decreases for increasing values of m and decreases for increasing values of λ.
2. Table 2 depicts the values of m for different values of α for fixed $\lambda = 0.5$. It is obvious that values of m increases as α decreases i.e. the ultimate strength capacity must increase if we wish to have a small tolerance level.

STRESS AND STRENGTH RELIABILITY

For the stress-strength model the probability $P = Pr(Y > X)$, when the random variable X and Y follows the pdfs (1.1) and (2.1), respectively is given by the following theorem.

Theorem 3.1 $P = Pr(Y > X)$ is given by

$$P = P(Y > X) = 1 - e^{-m(1-e^{-\lambda})} - \frac{1}{\left(m(1-e^{-\lambda})\right)^{a}}\gamma\left(a+1, m(1-e^{-\lambda})\right) \quad (3.1)$$

where $m = \beta\theta$

Proof:

$$P(Y > X) = \int_0^{\theta}\int_{y=x}^{\theta} f(x;\beta,\lambda)\,g(y;a,\theta)\,dydx \quad (3.2)$$

Since $m = \beta\theta \Longrightarrow \theta = \frac{m}{\beta}$

$$\int_0^{\frac{m}{\beta}}\int_{y=x}^{\frac{m}{\beta}} f(x;\beta,\lambda)g(y;a,\theta)\,dydx$$

Substitute $y = vx$ in (3.2), we have

$$= \int_0^{\frac{m}{\beta}} \int_{y=vx}^{\frac{m}{\beta x}} f(x;\beta,\lambda) g(y;a,\theta)\, x dv dx$$

$$= \int_0^{\frac{m}{\beta}} \int_{y=vx}^{\frac{m}{\beta x}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \left(\frac{a}{\theta}\right)\left(\frac{vx}{\theta}\right)^{a-1} x dv dx$$

$$= \int_0^{\frac{m}{\beta}} \int_{y=vx}^{\frac{m}{\beta x}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \left(\frac{a}{\theta}\right)\left(\frac{vx}{\theta}\right)^{a-1} x dv dx$$

$$= \int_0^{\frac{m}{\beta}} \int_{y=vx}^{\frac{m}{\beta x}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \left(\frac{a}{\theta}\right)\frac{v^{a-1}x^{a-1}}{\theta^{a-1}} x dv dx$$

$$= \int_0^{\frac{m}{\beta}} \int_{y=1}^{\frac{m}{\beta x}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{ax^a}{\theta^a} v^{a-1} dv dx$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{ax^a}{\theta^a}\left(\int_{y=1}^{\frac{m}{\beta x}} v^{a-1} dv\right) dx$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{ax^a}{\theta^a}\left[\frac{v^a}{a}\right]_1^{\frac{m}{\beta x}} dx$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{x^a}{\theta^a}\left[\left(\frac{m}{\beta x}\right)^a - 1\right]$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{x^a}{\theta^a}\left[\left(\frac{m}{\beta x}\right)^a - 1\right] dx$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} \frac{x^a}{\theta^a}\left(\frac{m}{\beta x}\right)^a dx$$

$$- \frac{1}{\theta^a}\int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda}) e^{-\beta x(1-e^{-\lambda})} x^a$$

$$= \int_0^{\frac{m}{\beta}} \beta(1-e^{-\lambda})\, e^{-\beta x(1-e^{-\lambda})} dx - \frac{1}{\theta^a}\int_0^{\frac{m}{\beta}} x^a \beta(1-e^{-\lambda})\, e^{-\beta x(1-e^{-\lambda})} dx \qquad (3.3)$$

Now, substituting $\beta x(1-e^{-\lambda}) = u$ in (3.3), we get

$$= \int_0^{m(1-e^{-\lambda})} \beta(1-e^{-\lambda}) e^{-u} \frac{du}{\beta(1-e^{-\lambda})e^{-u}}$$

$$- \frac{1}{\theta^a}\int_0^{m(1-e^{-\lambda})} u^{a+1-1}\, e^{-u} du$$

Where incomplete gamma function is given as $\gamma(a,z) = \int_0^z t^{a-1}e^{-t}dt$

$$P(Y > X) = 1 - e^{-m(1-e^{-\lambda})} - \frac{1}{\left(m(1-e^{-\lambda})\right)^a}\gamma\left(a+1, m(1-e^{-\lambda})\right).$$

This completes the proof.

Table 3: Strength-reliability of an item for selected values of m and a for $\lambda = 0.5$

m→ a↓	8	9	10	11	12	13	14
0.5	0.8638	0.9126	0.9447	0.9655	0.9787	0.987	0.992
1	0.8649	0.9109	0.9419	0.9626	0.9761	0.9849	0.9905
1.5	0.8818	0.9213	0.9479	0.9657	0.9776	0.9855	0.9907
2	0.9024	0.9351	0.9568	0.9713	0.981	0.9875	0.9918
2.5	0.9202	0.9472	0.9649	0.9766	0.9845	0.9897	0.9931
3	0.9335	0.9562	0.9709	0.9806	0.9871	0.9914	0.9942
3.5	0.9425	0.9622	0.9749	0.9833	0.9888	0.9925	0.995
4	0.9483	0.9659	0.9773	0.9849	0.9899	0.9932	0.9954
4.5	0.9519	0.9681	0.9788	0.9858	0.9905	0.9936	0.9957
5	0.954	0.9694	0.9795	0.9863	0.9908	0.9938	0.9958
5.5	0.9553	0.9701	0.98	0.9865	0.9909	0.9939	0.9959
6	0.9561	0.9705	0.9802	0.9867	0.991	0.9939	0.9959

DISCUSSION

While manufacturing an item, if the strength of an item follows Power function distribution, it is likely that the possible values of θ may have an upper limit say θ_0.For example, the capacity of accelerating an engine must be subject to maximum possible speed. For a fixed tolerance level α, suppose θ_α is the desired value of θ. In case $\theta_\alpha < \theta_0$, we may obtain the required value of $'a'$ and $'a_\alpha'$, by using Table 3, so that the item is manufactured with the strength distribution having parameters $(a_\alpha, \theta_\alpha)$ and consequently the desired strength reliability is achieved. However, if $\theta_\alpha > \theta_0$, we have to either adjust α or look for alternate item.

An Illustrative Example

Without loss of generality, we can be assumed that $\beta = 1$, so that $m = \theta$. Now suppose the maximum possible value of θ is 14. For $\alpha \leq 0.01$, we must have $m = \theta \geq 5$. Since θ cannot exceed 14, we have the option of finding the item in such way that $5 \leq \theta \leq 14$ and the corresponding value of a leads to maximum of P(Y>X). The cost factor of adjusting the parameter may take into consideration here as the cost of varying θ and $'a'$ may be different. Theoretically, the cost may be increasing or decreasing function of θ and $'a'$ depending upon the nature of the parameter. In our case $E(Y) = \frac{a\theta}{(a+1)}$ implies that the mean strength increase by increasing either of two parameters. Hence, we may assume the cost to be an increasing function of the respective parameters. Assuming the costs to be directly proportional to the required values of the parameters, the problem may be formulated as follows:

Let C_1 be the cost of adjusting one unit of a and C_2 be the cost of adjusting one unit of θ. Then our objective function may be of the following form

Minimize $C = C_1 a + C_2 \theta$ subject to $5 \leq \theta \leq 14$ and $P(Y > X) \geq 0.99$.

The problem may be solved analytically as follows:

Look into table 3 for $\theta = 12, 13$ *and* 14 and find those values of $'a'$ for which $P(Y > X) \geq 0.99$. Evaluate the cost function for each pair of (a, θ):

Table 4: Table for obtaining the optimum cost of manufacturing item

a	θ	$C = C_1 a + C_2 \theta$
4.5	12	$C = 4.5C_1 + 12C_2$
5	12	$C = 5C_1 + 12C_2$
5.5	12	$C = 5.5C_1 + 12C_2$
6	12	$C = 6C_1 + 12C_2$
6.5	12	$C = 6.5C_1 + 12C_2$
3	13	$C = 3C_1 + 13C_2$
3.5	13	$C = 3.5C_1 + 13C_2$
4	13	$C = 4C_1 + 13C_2$
4.5	13	$C = 4.5C_1 + 13C_2$
5	13	$C = 5C_1 + 13C_2$
5.5	13	$C = 5.5C_1 + 13C_2$
6	13	$C = 6C_1 + 13C_2$
6.5	13	$C = 6.5C_1 + 13C_2$
0.5	**14**	$\boldsymbol{C = 0.5C_1 + 14C_2}$
1	14	$C = C_1 + 14C_2$
1.5	14	$C = 1.5C_1 + 14C_2$
2	14	$C = 2C_1 + 14C_2$
2.5	14	$C = 2.5C_1 + 14C_2$
3	14	$C = 3C_1 + 14C_2$
3.5	14	$C = 3.5C_1 + 14C_2$
4	14	$C = 4C_1 + 14C_2$
4.5	14	$C = 4.5C_1 + 14C_2$
5	14	$C = 5C_1 + 14C_2$
5.5	14	$C = 5.5C_1 + 14C_2$
6	14	$C = 6C_1 + 14C_2$
6.5	14	$C = 6.5C_1 + 14C_2$

Clearly, the minimum of the cost lies at $C = 0.5C_1 + 14C_2$ depending upon the numerical values of C_1 and C_2.

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Source of Support: None Declared
Conflict of Interest: None Declared