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Appell transforms associated with L^2- expansions in terms of generalized heat polynomials

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Research Article

Abstract: The main purpose of this paper is to characterize functions which have L^2 expansions in terms of polynomial solutions $P_{n,\alpha,\beta}(x,t)$ of the generalized heat equation $\left[\frac{\partial^2}{\partial x^2}+\frac{4\alpha}{x}\frac{\partial}{\partial x}\right]u(x,t)=\frac{\partial}{\partial t}u(x,t)$, (A)

and in terms of the Appell transforms $W_{n,\alpha,\beta}(x,t)$ of the $P_{n,\alpha,\beta}(x,t)$. H^* denotes the C^2 class of functions $u(x,t)$ which, for $a<t<b$, satisfy (A) and for which

$$u(x,t)=\int_0^\infty G(x,y;\,t-t')\,u(y,t')\,d\mu(y),$$
$$d\mu(x)=2^{-(\alpha-\beta)}\left[\Gamma(3\alpha+\beta)\right]^{-1}x^{4\alpha}\,dx,$$

for all t,t', $a<t'<t<b$, the integral converging absolutely; where $G(x,y;t)$ is the source solution of (A).

***Keywords*:** Appell transforms, heat polynomial, Huygens property.

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1. Introduction and preliminary results: The generalized heat polynomial $P_{n,\alpha,\beta}(x,t)$ is a polynomial defined by

$$P_{n,\alpha,\beta}(x,t)=\sum_{k=0}^{n}2^{2k}\binom{n}{k}\frac{\Gamma(3\alpha+\beta)}{\Gamma(3\alpha+\beta+n-k)}x^{2n-2k}\,t^k, \tag{1.1}$$

$\alpha-\beta+\frac{1}{2}$ a fixed positive number. Note that when $\alpha=0$, $\beta=\frac{1}{2}$, $P_{n,o,\frac{1}{2}}=v_{2n}(x,t)$, the ordinary heat polynomials defined in [7, p.222].

For $t>0$, $P_{n,\alpha,\beta}(x,t)$ has the following integral representation.

$$P_{n,\alpha,\beta}(x,t)=\int_0^\infty y^{2n}\,G(x,y;\,t)\,d\mu(y), \tag{1.2}$$
$$d\mu(y)=2^{-(\alpha-\beta)}\left[\Gamma(3\alpha+\beta)\right]^{-1}x^{4\alpha}\,dx.$$

We may readily verify for $-\infty<x,\ t<\infty$, that $P_{n,\alpha,\beta}(x,t)$ satisfy the generalized heat equation

$$\Delta_x\,u(x,t)=\frac{\partial}{\partial t}u(x,t), \tag{1.3}$$

where

$$\Delta_x f(x) = f''(x) + \frac{4\alpha}{x} f'(x).$$

We denote by H the class of all C^2 functions which satisfy (1.3). The source solution of (1.3) is given by $G(x;t)$, where

$$G(x,y;t) = \left(\frac{1}{2t}\right)^{3\alpha+\beta} e^{-\left(\frac{x^2+y^2}{4t}\right)} g\left(\frac{xy}{2t}\right), \qquad (1.4)$$

with

$$g(z) = C_{\alpha,\beta}\, z^{-(\alpha-\beta)} I_{\alpha-\beta}(z),\; C_{\alpha,\beta} = 2^{-(\alpha-\beta)}\Gamma(3\alpha+\beta),$$

$I_r(z)$ being the Bessel function of imaginary argument of order r, and where
$G(x;t) = G(x,0;t)$ (Sec [1] for details).

Corresponding to the generalized heat polynomial $P_{n,\alpha,\beta}(x,t)$, its Appell transform $W_{n,\alpha,\beta}(x,t)$ is defined by

$$W_{n,\alpha,\beta}(x,t) = G(x,t) P_{n,\alpha,\beta}\left(\frac{x}{t}, -\frac{1}{t}\right), t > 0,\; n = 0,1,2,\ldots.., \qquad (1.5)$$

which is also a solution of (1.3). It follows from the definition of $P_{n,\alpha,\beta}(x,t)$ that

$$W_{n,\alpha,\beta}(x,t) = t^{-2n} G(x,t) P_{n,\alpha,\beta}(x,-t),\; t > 0,\; n = 0,1,2,\ldots.. \qquad (1.6)$$

Interesting fact about $P_{n,\alpha,\beta}(x,t)$ and $W_{n,\alpha,\beta}(x,t)$ is that they form a bioorthogonal system on $0 \le x < \infty$ .For $t > 0$, we have

$$\int_0^\infty W_{n,\alpha,\beta}(x,t) P_{m,\alpha,\beta}(x,-t)\, d\mu(x) = \frac{1}{b_n}\delta_{mn}, \qquad (1.7)$$

where

$$b_n = \Gamma(3\alpha+\beta) / \left[2^{4n}\, n!\; \Gamma(3\alpha+\beta+n)\right]. \qquad (1.8)$$

A consequence of (1.7) is a fundamental generating function for the bioorthogonal set $P_{n,\alpha,\beta}(x,-t)$, $W_{n,\alpha,\beta}(x,t)$. We have, for $0 \le x, y < \infty,\;\; -s < t < s,\;\; s > 0,$

$$G(x,y;\, s+t) = \sum_{n=0}^{\infty} b_n W_{n,\alpha,\beta}(y,s) P_{n,\alpha,\beta}(x,t). \qquad (1.9)$$

2. Inversion: For $t > s$, Set

$$K(x,y;s,t) = \sum_{n=0}^{\infty} b_n \left(\frac{t}{s}\right)^{\left(\frac{3\alpha+\beta}{2}\right)} e^{x^2/8t - y^2/8s} W_{n,\alpha,\beta}(x,t) P_{n,\alpha,\beta}(y,-s),$$

where b_n is defined by (1.8). Then as a consequence of the definitions and of (1.9), we have

$$K(x,y;s,t) = \left(\frac{t}{s}\right)^{\left(\frac{3\alpha+\beta}{2}\right)} e^{-\left(x^2(t-s)\right)/\left(8t(t+s)\right)} G\left(x\sqrt{2s/(t+s)},\;\; y\sqrt{(t+s)/2s}\,; t-s\right). \qquad (2.2)$$

Following [1; § 4], we have the following results:

Lemma 2.1:

(i) $K(x,y;s,t) \ge 0,\;\; 0 \le x, y < \infty,\; s < t,$ (2.3)

(ii) $\lim_{y\to\infty} K(x,y;s,t) = 0$, $0 \le x < \infty,\; s < t,$ (2.4)

(iii) $\lim_{s\to t^-} K(x, y; s, t) = 0$ uniformly $0 \le x, y < \infty,$ (2.5)

$|y - x| \ge \delta > 0,\ \delta$ any fixed positive number.

(iv) For x fixed , $0 \le x < \infty$,

$$\lim_{s\to t_-}\int_a^b K(x, y; s, t)\, d\mu(y) = 1,\ 0 \le a < x < b \le \infty, \tag{2.6}$$

$$= 0,\ 0 \le a \le b < x < \infty,$$

$$= 0,\ 0 \le x < a < b \le \infty.$$

It is now easy to establish the following fundamental inversion theorem.

Theorem 2.2: If $\phi \in L^1(0, \infty)$ and is continuous at x, then

$$\lim_{s\to t_-}\int_0^\infty K(x, y; s, t)\, \phi(y)\, d\mu(y) = \phi(x). \tag{2.7}$$

3. The Huygens property: A function $u(x, t)$ is said to have the Huygens property for $a < t < b$ if and only if $u(x,t) \in H$ and for every $t, t',\ a < t' < t < b,$

$$u(x,t) = \int_0^\infty G(x, y;\, t - t')\, u(y, t')\, d\mu(y) \tag{3.1}$$

the integral converging absolutely. We denote the class of all functions with the Huygens property by H^*. Functions of class H^* have a complex integral representation as given in the following result.

Lemma 3.1: If $u(x,t) \in H^*,\ a < t < b,$ then for $a < t < t' < b,$

$$u(x,t) = \int_0^\infty G(ix, y; t' - t)\, u(iy, t')\, d\mu(y). \tag{3.2}$$

The fact that $P_{n,\alpha,\beta}(x,t) \in H^*$ for $-\infty < t < \infty,$ and $W_{n,\alpha,\beta} \in H^*$ for $0 < t < \infty$ enables us to conclude that certain integrals involving functions of H^* are constant. A general result we can find in [4], but we state here the specific forms required in this paper.

Theorem 3.2: If $u(x, -t) \in H^*$ for $0 < t < \infty,$ then

$$\int_0^\infty u(x, -t)\, W_{n,\alpha,\beta}(x,t)\, d\mu(x) \tag{3.3}$$

is a constant.

Theorem 3.3: If $u(x,t) \in H^*$ for $0 < t < \infty,$ then

$$\int_0^\infty u(ix,t)\, W_{n,\alpha,\beta}(x,t)\, d\mu(x) \tag{3.4}$$

is a constant.

Theorem 3.4: If $u(x,t) \in H^*$ for $0 < t < \infty,$ then

$$\int_0^\infty u(x,t)\, P_{n,\alpha,\beta}(x, -t)\, d\mu(x) \tag{3.5}$$

is a constant.

4. Criteria for L^2 - expansions: In this section we establish criteria for a function $u(x,t)$ so that the series $\sum_{n=0}^{\infty} a_n P_{n,\alpha,\beta}(x,-t)$ converges in mean, with weight functions $G(x,t)$, to $u(x,t)$.

Theorem 4.1: Let $u(x,t) \in H^*$ for $-\sigma \le t < 0$, and

$$u(x,t)\left[G(x,-t)\right]^{1/2} \in L^2$$

for $-\sigma \le t < 0, \quad 0 \le x < \infty$. Then for $-\sigma \le t < 0$,

$$\lim_{N\to\infty} \int_0^{\infty} G(x,-t)\left|u(x,t) - \sum_{n=0}^{N} a_n \; P_{n,\alpha,\beta}(x,-t)\right|^2 d\mu(x) = 0 \tag{4.1}$$

and

$$\int_0^{\infty} G(x,-t)\left|u(x,t)\right|^2 d\mu(x) = \sum_{n=0}^{\infty} \frac{\left|a_n\right|^2}{b_n} t^{2n}, \tag{4.2}$$

where b_n is given by (1.8) and

$$a_n = b_n \int_0^{\infty} u(y,t) W_{n,\alpha,\beta}(y,-t)\, d\mu(y). \tag{4.3}$$

Proof: For fixed t, let $\phi(x,t)$ be a continuous function vanishing out-side a finite interval and such that, for $\varepsilon > 0$,

$$\int_0^{\infty} \left|u(x,-t)\left[G(x,t)\right]^{1/2} - \phi(x,t)\right|^2 d\mu(x) < \varepsilon, \; 0 < t \le \sigma. \tag{4.4}$$

Now set

$$\psi_n(x,t) = P_{n,\alpha,\beta}(x,-t)\left[G(x,t)\right]^{1/2}, \; 0 < t \le \sigma. \tag{4.5}$$

Then by (2.1), we have

$$K(x,y;s,t) = \sum_{n=0}^{\infty} b_n t^{-2n} \psi_n(x,t) \psi_n(y,s), \tag{4.6}$$

where b_n is defined by (1.8). Hence

$$\int_0^{\infty} K(x,y;\, s,t)\, \phi(y,t)\, d\mu(y) = \int_0^{\infty} \phi(y,t)\, d\mu(y) \sum_{n=0}^{\infty} b_n t^{-2n} \psi_n(x,t) \psi_n(y,s)$$

$$= \sum_{n=0}^{\infty} b_n t^{-2n} \psi_n(x,t) \int_0^{\infty} \psi_n(y,s)\, \phi(y,t)\, d\mu(y).$$

If we set

$$A_n(t) = b_n t^{-2n} \int_0^{\infty} \psi_n(y,t)\, \phi(y,t)\, d\mu(y), \tag{4.7}$$

and apply Theorem 2.2, we find that

$$\sum_{n=0}^{\infty} A_n(t) \psi_n(x,t) = \lim_{s\to t^-} \int_0^{\infty} K(x,y;s,t) \phi(y,t)\, d\mu(y) = \phi(x,t). \tag{4.8}$$

If we multiply both sides of (4.8) by $\phi(x,t)\, d\mu(x)$ and integrate between 0 and ∞, we obtain

$$\sum_{n=0}^{\infty} A_n(t) \int_0^{\infty} \psi_n(x,t)\, \phi(x,t)\, d\mu(x) = \int_0^{\infty} \phi^2(x,t)\, d\mu(x),$$

or by (4.7),

$$\sum_{n=0}^{\infty} \frac{t^{2n}}{b_n} A_n^2(t) = \int_0^{\infty} \phi^2(x,t)\, d\mu(x). \tag{4.9}$$

Now, let

$$c_n(t) = b_n t^{-2n} \int_0^{\infty} u(y,-t) \left[G(y,t)\right]^{1/2} \psi_n(y,t)\, d\mu(y). \tag{4.10}$$

Consider

$$I = \int_0^{\infty} \left\{ u(x,-t)\left[G(x,t)\right]^{1/2} - \sum_{k=0}^{n} C_k(t) \psi_k(x,t) \right\}^2 d\mu(x). \tag{4.11}$$

Since by (1.7), we have

$$\int_0^{\infty} \psi_n(x,t) \psi_m(x,t)\, d\mu(x) = \frac{t^{2n}}{b_n} \delta_{mn}, \tag{4.12}$$

with b_n given in (1.8), it follows that

$$I = \int_0^{\infty} \left[u(x,-t)\right]^2 G(x,t)\, d\mu(x) - \sum_{k=0}^{n} c_k^2(t) \frac{t^{2k}}{b_k}$$
$$\leq \int_0^{\infty} \left[u(x,-t)\right]^2 G(x,t)\, d\mu(x) + \sum_{k=0}^{n} \frac{t^{2k}}{b_k} \left[A_k(t) - c_k(t)\right]^2 - \sum_{k=0}^{n} c_k^2(t) \frac{t^{2k}}{b_k}$$
$$= \int_0^{\infty} [u(x,-t)]^2 G(x,t)\, d\mu(x) + \sum_{k=0}^{n} \frac{t^{2k}}{b_k} A_k^2(t) - 2\sum_{k=0}^{n} \frac{t^{2k}}{b_k} A_k(t) c_k(t)$$
$$= \int_0^{\infty} \left\{ u(x,-t)\left[G(x,t)\right]^{1/2} - \sum_{k=0}^{n} A_k(t) \psi_k(x,t) \right\}^2 d\mu(x)$$
$$\leq 2\int_0^{\infty} \left\{ u(x,-t)\left[G(x,t)\right]^{1/2} - \phi(x,t) \right\}^2 d\mu(x)$$
$$+ 2\int_0^{\infty} \left\{ \phi(x,t) - \sum_{k=0}^{n} A_k(t) \psi_k(x,t) \right\}^2 d\mu(x).$$

By (4.4), we have

$$I < 2\varepsilon + 2\int_0^{\infty} \phi^2(x,t)\, d\mu(x) + 2\int_0^{\infty} \sum_{k=0}^{n} A_k^2(t) \psi_k^2(x,t)\, d\mu(x) - 4\int_0^{\infty} \phi(x,t)\, d\mu(x) \sum_{k=0}^{n} A_k(t) \psi_k(x,t)$$
$$< 2\varepsilon + 2\int_0^{\infty} \phi^2(x,t)\, d\mu(x) + 2\sum_{k=0}^{n} A_k^2(t) \frac{t^{2n}}{b_n} - 4\sum_{k=0}^{n} A_k(t) \int_0^{\infty} \phi(x,t) \psi_k(x,t)\, d\mu(x)$$
$$< 2\,\varepsilon + 2\left\{ \int_0^{\infty} \phi^2(x,t)\, d\mu(x) - \sum_{k=0}^{n} A_k^2(t) \frac{t^{2n}}{b_n} \right\}.$$

It follows, therefore, by (4.9), that if n is sufficiently large, $I < 4\varepsilon$.

Thus

$$\lim_{N\to\infty} \int_0^\infty \left| u(x,-t)\left[G(x,t)\right]^{1/2} - \sum_{k=0}^{N} c_k(t)\psi_k(x,t) \right|^2 d\mu(x) = 0, \tag{4.13}$$

or by (4.5), we have (4.1) with $c_k(t)=a_k$. Theorem 3.4 establishes the fact that a_k is independent of t. Parseval's equation (4.2) follows since

$$\int_0^\infty G(x,t)\left|u(x,-t)\right|^2 d\mu(x) = \int_0^\infty \left| \sum_{n=0}^{\infty} c_n(t)\psi_n(x,t) \right|^2 d\mu(x)$$

$$= \sum_{n=0}^{\infty} \left|a_n\right|^2 \frac{t^{2n}}{b_n},$$

with the last equality a result of (4.12). Thus proof is completed.

An example illustrating the theorem is given by $u(x,t) = e^{a^2 t} f(ax)$.

This function satisfies the hypothesis for $-\infty < t < 0$ and we find that

$$\int_0^\infty G(x,t) F^2(ax)\, e^{-2a^2 t}\, d\mu(x) = F\left(2a^2 t\right), \qquad 0 < t < \infty, \tag{4.14}$$

where as

$$\sum_{n=0}^{\infty} \left|a_n\right|^2 \frac{t^{2n}}{b_n} = \sum_{n=0}^{\infty} b_n \left(a^2 t\right)^{2n} 2^{4n} \tag{4.15}$$

$$= F\left(2a^2 t\right), \qquad 0 < t < \infty.$$

Since

$$a_n = b_n \int_0^\infty e^{-a^2 t} F(ay)\, W_{n,\alpha,\beta}(y,t)\, d\mu(y), \quad 0 < t < \infty \tag{4.16}$$

$$= (2a)^{2n} b_n.$$

Although, in this example, $u(x,t) \in H^*$ for $-\infty < t < \infty$, the expansion (4.1) does not hold in the extended strip. Note that in this case the requirement that $u(x,t)\left[G(x,-t)\right]^{1/2}$ be in L^2 fails for $0 < t < \infty$. A modification of Theorem 4.1 when $u(x,t) \in H^*$ for $0 < t \le \sigma$ is given by the following result.

Theorem 4.2: If $u(x,t) \in H^*$ for $0 < t \le \sigma$, and if

$$u(ix,t)\left[G(x,t)\right]^{1/2} \in L^2.$$

For each fixed t, $0 < t \le \infty$, $0 \le x < \infty$, then for $0 < t \le \sigma$,

$$\lim_{N\to\infty} \int_0^\infty G(x,t) \left| u(ix,t) - \sum_{n=0}^{N} a_n P_{n,\alpha,\beta}(x,-t) \right|^2 d\mu(x) = 0, \tag{4.17}$$

and

$$\int_0^\infty G(x,t)\left|u(ix,t)\right|^2 d\mu(x) = \sum_{n=0}^{\infty} \frac{t^{2n}}{b_n} \left|a_n\right|^2, \tag{4.18}$$

where b_n is given by (1.8) and

$$a_n = b_n \int_0^\infty u(ix,t)\; W_{n,\alpha,\beta}\, d\mu(x), \quad 0 < t \le \sigma. \tag{4.19}$$

Proof: As in the preceding proof, we have

$$\lim_{N\to\infty}\int_0^\infty \left|u(ix,t)\left[G(x,t)\right]^{1/2}-\sum_{n=0}^{N}c_n(t)\psi_n(x,t)\right|^2 d\mu(x)=0,$$

with

$$c_n(t)=b_n t^{-2n}\int_0^\infty u(iy,t)\left[G(y,t)\right]^{1/2}\psi_n(y,t)\,d\mu(y).$$

Thus (4.17) holds with $c_n(t)=a_n$, which by Theorem 3.5, is independent of t. Further,

$$\int_0^\infty G(x,t)\left|u(ix,t)\right|^2 d\mu(x)=\int_0^\infty\left|\sum_{n=0}^{\infty}c_n(t)\psi_n(x,t)\right|^2 d\mu(x)$$

$$=\sum_{n=0}^{\infty}\frac{t^{2n}}{b_n}\left|a_n\right|^2$$

which is the Parseval equation (4.18). This completes the proof.

The example of the preceding theorem satisfies these hypotheses for $0<t<\infty$, and we have for $0<t<\infty$,

$$\int_0^\infty G(x,t)\,e^{2a^2t}F^2(i\,ax)\,d\mu(x)=F\left(2a^2t\right),$$

where as

$$a_n=b_n\int_0^\infty e^{a^2t}F(i\,ax)\,W_{n,\alpha,\beta}(x,t)\,d\mu(x),$$

so that

$$\sum_{n=0}^{\infty}\frac{t^{2n}}{b_n}\left|a_n\right|^2=F\left(2a^2t\right).$$

Criteria for expansions in terms of $W_{n,\alpha,\beta}(x,t)$ are given in the following result.

Theorem 4.3: If $u(x,t)\in H^*$ for $0<\sigma\le t$, and if

$$u(x,t)\left[G(ix,t)\right]^{1/2}\in L^2$$

for each fixed t, $0\le\sigma<t$, $0\le x<\infty$, then for $0<\sigma\le t$,

$$\lim_{N\to\infty}\int_0^\infty G(ix,t)\left|u(x,t)-\sum_{n=0}^{N}a_n W_{n,\alpha,\beta}(x,t)\right|^2 d\mu(x)=0, \tag{4.20}$$

where b_n is given by (1.8) and

$$a_n=b_n\int_0^\infty u(x,t)\,P_{n,\alpha,\beta}(x,-t)\,d\mu(x)\ ,\ \sigma\le t<\infty. \tag{4.22}$$

Proof: Again, as in Theorem 4.1 , since $u(x,t)\left[G(ix,t)\right]^{1/2}\in L^2$, we have

$$\lim_{N\to\infty}\int_0^\infty \left|u(x,t)\left[G(ix,t)\right]^{1/2}-\sum_{n=0}^{N}c_n(t)\psi_n(x,t)\right|^2 d\mu(x)=0, \tag{4.23}$$

with

$$c_n(t) = b_n t^{-2n} \int_0^\infty u(x,t)\left[G(ix,t)\right]^{1/2} \psi_n(x,t)\, d\mu(x). \qquad (4.24)$$

Now, (4.23) can be written in the form

$$\lim_{N\to\infty} \int_0^\infty G(ix,t) \left| u(x,t) - \sum_{n=0}^{N} c_n(t)(2t)^{3\alpha+\beta} t^{2n} W_{n,\alpha,\beta}(x,t) \right|^2 d\mu(x) = 0,$$

with (4.24) becoming

$$c_n(t) = b_n t^{-2n} (2t)^{-(3\alpha+\beta)} \int_0^\infty u(x,t) P_{n,\alpha,\beta}(x,-t)\, d\mu(x).$$

Hence if we set $a_n = c_n(t)\, t^{2n} (2t)^{3\alpha+\beta}$, a_n is independent of t, by Theorem 3.6, (4.20) is established. Moreover Parseval formula is

$$\int_0^\infty u(ix,t) \left| u(x,t) \right|^2 d\mu(x) = \sum_{n=0}^{\infty} \left| c_n^2(t) \right|^2 \frac{t^{2n}}{b_n} = \sum_{n=0}^{\infty} t^{-2n} (2t)^{-(\alpha+2\beta)} \frac{|a_n|^2}{b_n}.$$

Note that the function $u(x,t) = G(x,k;t)$ satisfies the conditions of the theorem for $0 < t < \infty$. In this case we have

$$a_n = b_n k^{2n},$$

and hence

$$\sum_{n=0}^{\infty} t^{-2n} (2t)^{-(6\alpha+2\beta)} \frac{|a_n|^2}{b_n} = \left(\frac{1}{2t}\right)^{6\alpha+2\beta} F\left(\frac{k^2}{2t}\right),$$

where as

$$\int_0^\infty G(ix,t) \left| G(x,k;t) \right|^2 d\mu(x) = \left(\frac{1}{2t}\right)^{6\alpha+2\beta} F\left(\frac{k^2}{2t}\right).$$

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