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Convexity in D-Metric Spaces and its Applications To Fixed Point Theorems

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Research Article

Abstract: Dhage introduced the concept of D-metric spaces. Following this Naidu, Rao, Srinivasa Rao and Asim, Aslam, Zafer discussed about the notion of balls in D-metric spaces. Wataru Takahashi introduced a convex structure in metric spaces and formulated some fixed point theorems for nonexpansive mappings. The purpose of this paper is to formulate some fixed point theorems in different convex structures in metric spaces.

Key words: Fixed points, D-metric and nonexpansive mappings.

MSC 2010: 47H10, 54E50

1. Introduction and Preliminaries

The concept of D-metric space has been investigated initially by Dhage[3], D-metric spaces are further studied by Naidu, Rao, Srinivasa Rao[4] and Asim, Aslam, Zafer[2]. In 1970 Wataru Takahashi introduced the concept of convex structure in metric spaces and established some fixed point theorems.

Following this the authors[5, 6] introduced the concept of E-convex structure in metric spaces and established some fixed point theorems. Further the authors[7] studied the concepts of Strong, Weak and Quasi convex structures in metric spaces and ultra metric spaces. In this paper we introduce the concepts of Strong, Weak and Quasi convex structures in D-metric spaces and discuss their basic properties. Further, we establish some fixed point theorems in these structures.

For the notations and terminology we refer **W**ataru Takahashi[9] and Shyamala Malini et al.[5, 6].

Definition 1.1: Let (X, d) be a metric space. A mapping $W: X\times X\times[0, 1]\to X$ is said to be a convex structure[9] on (X, d) if for each $(x, y, \lambda)\in X\times X\times[0, 1]$ and for all $u\in X$, the condition $d(u, W(x, y, \lambda))\le\lambda d(u,x)+(1-\lambda)d(u,y)$ holds. The metric space (X, d) together with the convex structure W is called a convex metric space denoted by (X, d, W).

The following definition is due to Dhage.

Definition 1.2: Let X be a nonempty set. Let $D: X\times X\times X\to[0, \infty)$ be a function satisfying the following conditions for all $x, y, z\in X$.

$D(x, y, z) = 0$ if and only if $x = y= z$.

$D(x, y, z)= D(\sigma y, \sigma x, \sigma z)$ for every permutation σ of x, y, z in X.

$D(x, y, z) \le D(x, y, a) + D(x, a, z) + D(a, y, z)$ for every $a\in X$.

Then the function D is called a D-metric[3] on X. The set X together with a D-metric D is called a D-metric space denoted by (X, D).

The notion of open balls in a metric space plays a dominant role in Analysis. Various Mathematicians extended the notion of this ball to D-metric spaces. The following definition is due to Dhage[3].

Definition 1.3: Let (X, ρ) be a D-metric space and $x_0\in X$ then for every r>0

$$S^*(x_0, r) = \{y\in X: \rho(x_0, y, y) < r\} \quad \textbf{Eqn. 4.1}$$

and $$S(x_0, r) = \bigcap_{z\in X}\{ y, z\in X: \rho(x_0, y, z) < r\} \quad \textbf{Eqn. 4.2}$$

The above definition was not appropriate and it was rectified by Naidu et al[11].

The next definition is due to Naidu et.al.

Definition 1.4: Let (X, ρ) be a D-metric space and $x_0\in X$ then for every r>0

$$S^*(x_0, r) = \{x\in X: \rho(x_0, x, x) < r\} \quad \textbf{Eqn. 4.3}$$

$$S(x_0, r) = x\in S^*(x_0, r):\rho(x_0, x, y) < r \text{ for all } y\in S^*(x_0, r)\} \quad \textbf{Eqn. 4.4}$$

$$\hat{S}(x_0, r) =\{x_0\}\cup\{x\in X: \sup_{y\in X}\rho(x_0, x, y) < r\} \quad \textbf{Eqn. 4.5}$$

Following this Asim, Aslam and Zafer[2] modified the above definition assuming that $\sup_{y\in X}\rho(x_0, x, y) < r$ is not always possible.

Definition 1.5: Let (X, ρ) be a D-metric space and $x_0\in X$ then for every r>0

$$S^*(x_0, r)= \{x\in X: \rho(x_0, x, x) < r\} \quad \textbf{Eqn. 4.6}$$

$S(x_0, r)=\{x_0\}\cup\{x\in X: \sup_{y\in S^*(x_0, r)} \rho(x_0, x, y)< r\}$ **Eqn. 4.7**

There is a uniformity in the definition of the ball $S^*(x_0, r)$, in[2],[3],[4]. However the definition for $S(x_0, r)$(Eqn. 4.2) is not meaningful as was said in[4][page no. 134, line 9].

Also Asim, Aslam and Zafer expressed their dissatisfaction in the definition of $\hat{S}(x_0, r)$(Eqn. 4.5), quoting that $\sup_{y\in S^*(x_0, r)} \rho(x_0, x, y)< r$ is not always possible for all $x\in X$. But their assumption is wrong. For even if $\sup_{y\in X} \rho(x_0, x, y)< r$ does not exist or if $\sup_{y\in S^*(x_0, r)} \rho(x_0, x, y)$ is not less than r for all $x\in X$ then also the definition is meaningful. More precisely if the above condition prevails for some x then we can take $\hat{S}(x_0, r) =\{x_0\}$.

Thus we have the following three types of open balls in D-metric spaces.

Definition 1.6: Let (X, ρ) be a D-metric space, $x_0\in X$ and $r>0$.

$S_1(x_0,r)=\{x\in S^*(x_0,r):\rho(x_0,x,y)<r \text{ for all } y\in S^*(x_0, r)\}$.(in the sense of Naidu)

$\hat{S}(x_0, r) =\{x_0\}\cup\{x\in X: \sup_{y\in X} \rho(x_0, x, y) < r\}\}$.(in the sense of Naidu)

$S_2(x_0, r)=\{x_0\}\cup\{x\in X: \sup_{y\in S^*(x_0, r)} \rho(x_0, x, y)< r \}\}$.(in the sense of Asim)

2. Convex D-Metric Spaces

We introduce the concept of convex structures in D-metric spaces and discuss its basic properties. Some fixed point theorems are also established in this structure.

Definition 2.1: Let (X, D) be a metric space. A mapping $W: X\times X\times X\times(0, 1]\to X$ is said to be a convex structure on (X, D) if for each $(x, y, z, \lambda)\in X\times X\times X\times(0, 1]$ and for all $u, v\in X$ the condition

$$D(u,v,W(x,y,z,\lambda))\le\frac{\lambda}{3}D(u,v,x)+\frac{\lambda}{3}D(u,v,y)+\frac{\lambda}{3}D(u,v,z) \text{ holds.}$$

If W is convex on a D-metric space (X, D), then the triplet (X, D, W) is called a convex D-metric space.

Example 2.2: Consider a linear space L which is also a D-metric space with the following properties:

(i) For $x, y, z\in L$, $D(x, y, z) = D(x–y–z, 0, 0)$;

(ii) For $x, y, z \in L$ and $\lambda\in (0, 1]$,

$$D(\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3},0,0)\le \frac{\lambda}{3}D(x,0,0)+ \frac{\lambda}{3}D(y,0,0) + \frac{\lambda}{3}D(z, 0, 0).$$

Let $W(x, y, z, \lambda) = \frac{\lambda}{3}x + \frac{\lambda}{3}y + \frac{\lambda}{3}z$ for all $x, y, z\in X$ and $\lambda\in (0, 1]$. Then (L, D, W) is a convex D-metric space.

Definition 2.3: A subset M of convex D-metric space. (X, D, W) is said to be a convex set in (X, D, W) if $W(x, y, z, \lambda)\in M$ for all $x, y, z\in M$ and for all λ with $0<\lambda\le1$.

Proposition 2.4: If $\{K_\alpha:\alpha\in\Delta\}$ is a family of convex subsets of the convex D-metric space (X, D, W), then $\bigcap_{\alpha\in\Delta} K_\alpha$ is also a convex subset in (X, D, W).

Proposition 2.5: Let (X, D, W) be a convex D-metric space, $u\in X$, $r >0$. The balls $S_1(u, r)$ (in the sense of Naidu), $\hat{S}(u, r)$ (in the sense of Naidu) and $S_2(u, r)$ (in the sense of Asim) in (X, D, W) are convex subsets of (X, D, W). The balls $S^*(u, r)$ and $S^*[u, r]$ are also convex in (X, D, W).

Definition 2.6: Let (X, D, W) be a convex D-metric space. Let $A\subseteq X$.

$R_x (A) = \sup\{ D(x, y, y): y\in A\}$.

$R(A) = \inf\{ R_x(A): x\in A\}$.

$A_c =\{x\in A: R_x(A) = R(A)\}$.

Diameter of $A =\delta(A)=\sup\{D(x, y, y): x, y\in A\}$.

Definition 2.7: Let (X, D, W) be a convex D-metric space. Let A be a subset of X. A point $x\in A$ is a diametral point of A if $\sup_{y\in A} D(x, y, y) = \delta(A)$.

Definition 2.8: A convex D-metric space (X, D, W) is said to have the Property(DC) if every bounded decreasing sequence of nonempty closed convex subsets of (X, D, W) has nonempty intersection.

Proposition 2.9 Let (X, D, W) be a convex D-metric space. Let $A\subseteq X$. If (X, D, W) has the Property(DC). Then A_c is nonempty, closed and convex.

Proof: Let $x\in A$. For each positive integer n, define $A_n(x)=\{y\in A:D(x,y,x)\le R(A)+\frac{1}{n}\}$and $C_n= \bigcap_{x\in A} A_n(x)$.

Since $x\in A_n(x)$, $A_n(x)$ is nonempty and closed. Let $z\in A_n(x)$ and $\lambda\in (0, 1]$.

$$D(x,x,W(x, y, z,\lambda))\le\frac{\lambda}{3}D(x, x, x)+\frac{\lambda}{3}D(x,x,y)+\frac{\lambda}{3}D(x,x,z)$$

$\leq \frac{\lambda}{3}(R(A)+\frac{1}{n})+\frac{\lambda}{3}(R(A)+\frac{1}{n})+\frac{\lambda}{3}(R(A)+\frac{1}{n})$

$<(R(A)+\frac{1}{n})$.

Therefore $W(x, y, z,\lambda)\in A_n(x)$ which implies that $A_n(x)$ is a convex set.

We claim that $\{C_n\}$ is a bounded decreasing sequence of nonempty, closed and convex subsets.

Let $\varepsilon > 0$. Then by the Definition of R(A),there exists $y\in A$ with $R_y(A)\leq R(A)+\varepsilon$ and by the Definition of $R_y(A)$, $D(x, y, y) \leq R_y(A)$ for every $x\in A$.

$<R(A)+\varepsilon$ for every $x\in A$

As ε is arbitrary for every $x\in A$, $D(x, y, y)\leq R(A)+\varepsilon< R(A)+\frac{1}{n}$. Hence $y\in A_n(x)$ for every $x\in A$ and hence $y\in \bigcap_{x\in A} A_n(x) = C_n$. Thus C_n is nonempty. Since each A_n is a closed set C_n is also closed. By Proposition 2.5, C_n is a convex set. Hence C_n is a nonempty, closed and convex set for all n.

Let $z\in C_{n+1}$ .Then $z\in A_{n+1}(x)$ for every $x\in X$, $D(x, x, z)\leq R(A)+\frac{1}{n+1}\leq R(A)+\frac{1}{n}$. Therefore $z\in A_n(x)$ for every $x\in X$ which implies that $z\in C_n$. Now we prove that that $A_c=\bigcap_{n=1}^{\infty} C_n$. Since the space (X, D, W) has the Property (DC), $\bigcap_{n=1}^{\infty} C_n$ is nonempty. Let $x\in \bigcap_{n=1}^{\infty} C_n$. Then $x\in A_n(y)$ for every $y\in A$ and for every n. Now $D(x, y, y)\leq R(A)+\frac{1}{n}$ for every $y\in A$ and for every n. Thus $R_x(A)\leq R(A)$. By the Definition of R(A), we have $R(A)\leq R_x(A)$. Hence $R(A)=R_x(A)$ implies $x\in A_c$.

Conversely, let $x\in A_c$. Then $R_x(A)=R(A)$. Now, for every $y\in A$ $D(x, y, y)\leq R_x(A)=R(A)\leq R(A)+\frac{1}{n}$ for every n. This shows that $x\in A_n(y)$ for every $y\in X$ and for every n. This completes the proof.

Proposition 2.10: Let M be a nonempty compact subset of a convex D-metric space (X, D, W) and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive,then there exists an element $u\in K$ such that $\sup\{D(y, y, u):x, y\in M\}<\delta(M)$.

Proof: Since M is compact, we may find $x_1, x_2, x_3\in M$ such that $D(x_1, x_2, x_3) = \delta(M)$. Let $M_0 \subseteq M$ be maximal so that $M_0 \supseteq\{x_1, x_2, x_3\}$ and $D(x, y, y) = 0$ or $\delta(M)$ for all $x, y\in M_0$. Since M is compact, M_0 is finite. Let us assume that $M_0=\{x_1,x_2,\ldots,x_n\}$.

Let $y_1 = W(x_1, x_2, x_3, \frac{1}{2})$

$y_2 = W(x_3, x_4\, y_1, \frac{1}{3})$

….

$y_{n-2} = W(x_{n-2}, x_{n-1}, y_{n-3}, \frac{1}{n-1})$

$y_{n-1}= W(x_{n-1}, x_n, y_{n-2}, \frac{1}{n}) = u$.

As K is a convex set, $u\in K$. Since M is compact, we can find $y_0\in M$ such that $D(y_0, y_0,u)= \sup\{D(y, y, u): y\in M\}$. Now by the Definition 2.1, we have

$D(y_0, y_0, u) = D(y_0, y_0, W(x_{n-1}, x_n, y_{n-2}, \frac{1}{n})$

$\leq \frac{\lambda}{3}D(y_0, y_0,x_{n-1}) +\frac{\lambda}{3}D(y_0, y_0, x_n) +\frac{\lambda}{3} D(y_0, y_0, y_{n-2})$

$=\frac{\lambda}{3}D(y_0,y_0,x_{n-1}) +\frac{\lambda}{3}D(y_0,y_0,x_n)+$

$\frac{\lambda}{3}D(y_0,y_0,W(x_{n-2}, x_{n-1}, y_{n-3}; \frac{1}{n-1}))$

$\leq\frac{\lambda}{3}\sum_{k=1}^{n} D(y_0, y_0, x_k) \leq\delta(M)$.

Suppose $D(y_0, y_0, u) = \delta(M)$. Then $D(y_0, y_0, x_k) = \delta(M) >0$ for all k=1,2,3…,n. Since $y_0\in M_0$ we have $y_0= x_k$ for some k=1,2,…,n. This implies $\delta(M) =0$ which is a contradiction. Therefore $\sup\{D(x, y, y): x\in M\}=D(y_0, y_0,u) <\delta(M)$.

Definition 2.11:A convex D-metric space (X, D, W) is said to have normal structure if for each closed bounded convex subset A of (X, D, W) which contains at least two points, there exists $x\in A$ which is not a diametral point of A.

Definition 2.12: Let (X, D, W) be a convex D-metric space and K be a subset of (X, D, W). A mapping T of K into X is said to be ***nonexpansive*** if for any three elements x, y and z of K, we have $D(T(x), T(y), T(z))\leq D(x, y, z)$.

Theorem 2.13: Let (X, D, W) be a convex D-metric space. Suppose that (X, D, W) has the Property(DC). Let K be a nonempty bounded closed convex subset of (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof: Let Φ be a family of all nonempty closed and convex subsets of K, such that each of which is mapped into itself by T. By the Property(DC) and Zorn's Lemma, Φ has a minimal element A. We show that A consists of a single point. Let $x\in A_c$. Since T is nonexpansive map, $D(T(x), T(y), T(y)))\leq D(x, y, y)$ for all $y\in A$.

Again for every $y\in A$, $D(x, y, y)\leq R_x(A)$, which implies $D(T(x), T(y), T(y))\leq D(x, y, y)\leq R_x(A) = R(A)$ for all $y\in A$ and hence $T(A)$ is contained in the closed ball $S^*[T(x),R(A)]$. Since $T(A)\cap S^*[T(x), R(A)])\subseteq A\cap S^*[T(x), R(A)]$ the minimality of A implies $A\subseteq S^*[T(x), R(A)]$. Hence $R_{T(x)}(A)\leq R(A)$. We know that $R(A)\leq R_{T(x)}$ (A) for all $x\in A$. Thus $R(A)=R_{T(x)}(A)$. Hence $T(x)\in A_c$ and $T(A_c)\subseteq A_c$. By using Proposition 2.9, $A_c\in \Phi$. If $z\in A_c$, then $D(y, y, z)\leq R_z(A) =R(A)$ that implies $\delta(A_c)\leq R(A)$. By the Definition of normal structure $R(A)<\delta(A)$. This proves that $\delta(A_c)\leq R(A)<\delta(A)$. This is a contradiction to the minimality of A. Therefore we must have $\delta(A)=0$ that implies A consists of single point.

Definition 2.14: Let K be a compact convex D-metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have invariant property in K if for any compact and convex subset A of K such that $TA\subseteq A$ for each $T\in \mathcal{F}$, there exists a compact subset $M\subseteq A$ such that $TM=M$ for each $T\in \mathcal{F}$.

Theorem 2.15: Let (X, D, W) be a convex D-metric space and K be a compact convex D-metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.

Proof: By applying Zorn's Lemma, we can find a minimal nonempty convex compact set $A\subseteq K$ such that A is invariant under each $T\in \mathcal{F}$. If A consists of a single point, then the theorem is proved. Suppose that A contains more than one point, then by the hypothesis, there exists a compact subset M of A such that $M= \{T(x): x\in M\}$ for each $T\in \mathcal{F}$.

If M contains more than one point, by Proposition 2.10, there exists an element u in the least convex set containing M such that $\rho = \sup\{D(x, x, u): x\in M\}< \delta(M)$, where $\delta(M)$ is the diameter of M.

Let us define $A_0 = \bigcap_{x\in M}\{y\in M: D(x, y, y)\leq \rho\}$.

Clearly A_0 is nonempty and closed. By using Proposition 2.4, A_0 is convex. Since u is not in A_0, A_0 is a proper subset of A invariant under each T in $\mathcal{F}$. This is a contradiction to the minimality of A.

3. Strong convex D-metric spaces

We introduce the concept of strong convex structure on D-metric spaces and extend some fixed point theorems of convex D-metric spaces to strong convex D-metric spaces.

Definition 3.1: Let (X, D) be a metric space. A mapping $W:X\times X\times X\times(0, 1]\rightarrow X$ is said to be a strong convex structure on (X, D) if for each $(x, y, z, \lambda)\in X\times X\times X\times(0, 1]$ and for all $u, v\in X$ the condition

$$D(u, v, W(x, y, z,\lambda))\leq\max\{ \frac{\lambda}{3}D(u, v, x), \frac{\lambda}{3}D(u, v, y), \frac{\lambda}{3}D(u, v, z)\} \text{ holds.}$$

If W is strong convex on a D-metric space (X, D), then the triplet (X, D, W) is called a strong convex D-metric space.

Example 3.2: Consider a linear space L which is also a D-metric space with the following properties:

(i) For $x, y, z\in L$, $D(x, y, z) = D(x- y- z, 0, 0)$;

(ii) For $x, y, z \in L$ and $\lambda\in (0, 1]$,

$$D(\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z, 0, 0)\leq\max\{ \frac{\lambda}{3}D(x,0,0) , \frac{\lambda}{3}D(y,0,0), \frac{\lambda}{3} D(z, 0, 0)\}.$$

Let $W(x, y, z, \lambda) =\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z$ for all $x, y, z\in X$ and $\lambda\in (0, 1]$. Then (L, D, W) is a strong convex D-metric space.

Proposition 3.3: If (X, D, W) is a strong convex D-metric space then it is a convex D-metric space.

Definition 3.4: A subset M of a strong convex D-metric space (X, D, W) is said to be a convex set (X, D, W) if $W(x, y, z, \lambda)\in M$ for all $x, y, z\in M$ and for all λ with $0<\lambda\leq1$.

Proposition 3.5: Let (X, D, W) be a strong convex D-metric space, $u\in X$, $r >0$. Then

(i) If $\{K_\alpha:\alpha\in \Delta\}$ is a family of convex subsets of the strong convex D-metric space (X, D, W), then $\bigcap_{\alpha\in\Delta}K_\alpha$ is also a convex subset in (X, D, W).

(ii) The balls $S_1(u, r)$.(in the sense of Naidu), $\hat{S}(u, r)$.(in the sense of Naidu) and $S_2(u, r)$ (in the sense of Asim) in (X, D, W) are convex subsets of (X, D, W). The balls $S^*(u, r)$ and $S^*[u, r]$ are convex in (X, D, W).

Proposition 3.6: Let (X, D, W) be a strong convex D-metric space. Let $A\subseteq X$. If (X, D, W) has the Property(DC), then A_c is nonempty, closed and convex.

Proof: Analogous to Proposition 2.9.

Proposition 3.7: Let M be a nonempty compact subset of a convex D-metric space (X, D, W) and let K be the least closed convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u\in K$ such that $\sup\{D(y, y, u):x, y\in M\}<\delta(M)$.

Proof: Since (X, D, W) is a strong convex D-metric space, by using Proposition 3.3, (X, D, W) is a convex D-metric space. Then K is also a convex set containing

M in the convex D-metric space (X, D, W). By applying Proposition 2.10, there exists an element $u \in K$ such that $\sup\{d(x, u): x \in M\} < \delta(M)$.

Definition 3.8: A strong convex D-metric space (X, D, W) is said to have normal structure if for each closed bounded convex subset A of (X, D, W) which contains at least two points, there exists $x \in A$ which is not a diametral point of A.

Theorem 3.9: Let (X, D, W) be a strong convex D-metric space. Suppose that (X, D, W) has the Property(DC). Let K be a nonempty bounded closed convex subset of (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof: Since (X, D, W) is a strong convex D-metric space, by using Proposition 3.3, (X, D, W) is a convex D-metric space. Then K is also a bounded closed convex subset of convex D-metric space (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then by applying Theorem 2.13, T has a fixed point in K.

Definition 3.10: Let K be a compact strong convex D-metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have invariant property in K if for any compact and strong convex subset A of K such that $TA \subseteq A$ for each $T \in \mathcal{F}$, there exists a compact subset $M \subseteq A$ such that TM=M for each $T \in \mathcal{F}$.

Theorem 3.11: Let (X, D, W) be a strong convex D-metric space. Let K be a compact strong convex D-metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.

Proof: Since (X, D, W) is a strong convex D-metric space, by using Proposition 3.3, (X, D, W) is a convex D-metric space. Then K is also a convex D-metric space (X, D, W). If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then by applying Theorem 2.15 the family has a common fixed point in K.

4. J-convex D-metric spaces

When maximum is replaced by minimum in the definition of strong D-convex structures we have the new convex structure called the J-convex structure. In this section we introduce the concept of J-convex structure and extend some properties of convex D-metric spaces and strong convex D-metric spaces to J-convex D-metric spaces.

Definition 4.1: Let (X, D) be a metric space. A mapping $W: X \times X \times X \times (0, 1) \to X$ is said to be a J-convex structure on a D-metric space (X, D) if for each $(x, y, z, \lambda) \in X \times X \times X \times I$ and for all $u, v \in X$ the condition $D(u, v, W(x, y, z, \lambda)) \leq \min\{\frac{\lambda}{3} D(u, v, x), \frac{\lambda}{3} D(u, v, y), \frac{\lambda}{3} D(u, v, z)\}$ holds.

If W is a J-convex on a D-metric space (X, D), then the triplet (X, D, W) is called a J-convex D-metric space.

Example 4.2: Consider a linear space L which is also a D-metric space with the following properties:

(i) For $x, y, z \in L$, $D(x, y, z) = D(x-y-z, 0, 0)$;

(ii) For $x, y, z \in L$ and $\lambda \in (0, 1)$,

$$D(\frac{\lambda}{3} x + \frac{\lambda}{3} y + \frac{\lambda}{3} z, 0, 0) \leq \min\{\frac{\lambda}{3} D(x,0,0), \frac{\lambda}{3} D(y,0,0), \frac{\lambda}{3} D(z, 0, 0)\}.$$

Let $W(x, y, z, \lambda) = \frac{\lambda}{3} x + \frac{\lambda}{3} y + \frac{\lambda}{3} z$ for all $x, y, z \in X$ and $\lambda \in (0, 1)$. Then (L, D, W) is a J-convex D-metric space.

Definition 4.3: A subset M of a J-convex D-metric space (X, D, W) is said to be a convex set if $W(x, y, z, \lambda) \in M$ for all $x, y, z \in M$ and for all λ with $0<\lambda<1$.

Proposition 4.4: Let (X, D, W) be a J-convex D-metric space, $u \in X$, $r > 0$. Then

(i) If $\{K_\alpha : \alpha \in \Delta\}$ is a family of convex subsets of the J-convex D-metric space (X, D, W), then $\bigcap_{\alpha \in \Delta} K_\alpha$ is also a convex subset in (X, D, W).

(ii) The balls $S_1(u, r)$.(in the sense of Naidu), $\hat{S}(u, r)$.(in the sense of Naidu) and $S_2(u, r)$ (in the sense of Asim) in (X, D, W) are convex subsets of (X, D, W). The balls $S^*(u, r)$ and $S^*[u, r]$ are convex in (X, D, W).

Definition 4.5: A J-convex D-metric space (X, D, W) is said to have the Property(JDC) if every bounded decreasing sequence of nonempty closed convex subsets of (X, D, W) has nonempty intersection.

Proposition 4.6: Let (X, D, W) be a J-convex D-metric space. Let $A \subseteq X$. If (X, D, W) has the Property(JDC), then A_c is nonempty, closed and convex.

Proof: Analogous to Proposition 2.9.

Proposition 4.7: Let M be a nonempty compact subset of a J-convex D-metric space (X, D, W) and let K be the least closed J-convex set containing M. If the diameter $\delta(M)$ is positive, then there exists an element $u \in K$ such that $\sup\{D(y, y, u): y \in M\} < \delta(M)$.

Proof: Analogous to Proposition 2.10.

Definition 4.8: A J-convex D-metric space (X, D, W) is said to have normal structure if it has the same normal structure if for each closed bounded J-convex subset A of (X, D, W) which contains at least two points, there exists x∈A which is not a diametral point of A.

Theorem 4.9: Let (X, D, W) be a J-convex D-metric space. Suppose that (X, D, W) has the Property(JDC). Let K be a nonempty bounded closed convex subset of (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof: Analogous to Proposition 2.13.

Definition 4.9: Let K be a compact J-convex D-metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have invariant property in K if for any compact and J-convex subset A of K such that TA⊆A for each T∈$\mathcal{F}$, there exists a compact subset M⊆A such that TM=M for each T∈$\mathcal{F}$.

Theorem 4.12: Let (X, D, W) be a J-convex D-metric space. Let K be a compact convex D-metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.

Proof: Analogous to Proposition 2.15.

5. Weak convex D-metric spaces

Definition 5.1: Let (X, D) be a D-metric space. A mapping W:X×X×X×(0, 1]→ X is said to be a weak convex structure on (X, D) if for each (x, y, z, λ)∈X×X×X×(0, 1] and for all u, v∈X the condition

D(u, v, W(x, y, z, λ))≤ D(u, v, x)+ D(u, v, y)+D(u, v, z) holds .

If W is a weak convex structure on (X, D), then the triplet (X, D, W) is called a weak convex D-metric space.

Example 5.2: Consider a linear space L which is also a D-metric space with the following properties:

(i) For x, y, z∈L, D(x, y, z) = D(x–y– z, 0, 0);

(ii) For x, y, z∈L and λ∈(0, 1],

$$D(\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z,0,0)\le D(x,0,0)+D(y,0,0)+D(z,0,0).$$

Let $W(x, y, z, \lambda)=\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z$ for all x, y, z∈X and λ∈(0, 1],. Then (L, D, W) is a weak convex D-metric space.

Proposition 5.3: Every convex D-metric space (X, D, W) is a weak convex D-metric space.

Definition 5.4: A subset M of weak convex D-metric space (X, D, W) is said to be a weak convex if W(x, y, z, λ)∈M for all x, y, z∈M and for all λ with 0<λ≤1.

Proposition 5.5: Let (X, D, W) be a weak convex D-metric space, u∈X, r >0. Then

(i) If $\{K_\alpha:\alpha\in\Delta\}$ is a family of weak convex subsets of the weak convex D-metric space (X, D, W), then $\bigcap_{\alpha\in\Delta}K_\alpha$ is also a weak convex subset in (X, D, W).

(ii) The balls $S_1(u, r)$.(in the sense of Naidu),$\hat{S}(u, r)$.(in the sense of Naidu) and $S_2(u, r)$ (in the sense of Asim) in (X, D, W) are weak convex subsets of (X, D, W). The balls S*(u, r) and S*[u, r] are weak convex in (X, D, W).

Definition 5.6: A weak convex D-metric space (X, D, W) is said to have the Property(WDC) if every bounded decreasing sequence of nonempty closed weak convex subsets of (X, D, W) has nonempty intersection.

Proposition 5.7: Let (X, D, W) be a weak convex D-metric space. Let A⊆X. If (X, D, W) has the Property(WDC), then A_c is nonempty, closed and weak convex.

Proof: Analogous to Proposition 2.9.

Theorem 5.8: Let M be a nonempty compact subset of a weak convex D-metric space (X, D, W) and let K be the least closedweak convex set containing M. If the diameter δ(M) is positive, then there exists an element u∈K such that sup{D(y, y, u): y∈M}<δ(M).

Proof: Analogous to Proposition 2.10.

Theorem 5.9: Let (X, D, W) be a weak convex D-metric space. Suppose that (X, D, W) has the Property(DC). Let K be a nonempty bounded closed weak convex subset of (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof: Analogous to Proposition 2.13.

Definition 5.10: Let K be a compact weak convex D-metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have invariant property in K if for any compact and weak convex subset A of K such that TA⊆ A for each T∈$\mathcal{F}$, there exists a compact subset M⊆A such that TM=M for each T∈$\mathcal{F}$.

Theorem 5.11: Let (X, D, W) be a weak convex D-metric space. Let K be a compact weak convex D-metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.

Proof: Analogous to Proposition 2.15.

6. Quasi convex D-Metric Spaces

A new convex structure in a metric space can be introduced by taking $\lambda=\frac{1}{3}$ in the definition of convex structure introduced by Wataru Takahashi. This new structure is named as quasi convex structure in D-metric spaces. In this section we introduce such a structure and establish some basic properties of quasi convex D-metric spaces.

Definition 6.1: Let (X, D) be a metric space. A mapping W: X×X×X×(0, 1]→X is said to be a Quasi convex structure on (X, D) if for each (x, y, z, λ)∈ X×X×X×(0, 1] and for all u, v∈ X the condition

$$D(u,v,W(x,y,z,\lambda))\leq \frac{1}{3}D(u,v,x)+\frac{1}{3}D(u,v,y)+\frac{1}{3}D(u,v,z).$$

If W is a Quasi convex on a D-metric space (X, D), then the triplet (X, D, W) is called a Quasi convex D-metric space.

Example 6.2: Consider a linear space L which is also a D-metric space with the following properties:

(i) For x, y, z∈ L, D(x, y, z) = D(x–y–z, 0, 0);

(ii) For x, y, z∈ L and λ∈ (0, 1].

$$D(\frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z,0,0)\leq \frac{1}{3}D(x,0,0)+\frac{1}{3}D(y,0,0)+\frac{1}{3}D(z, 0, 0).$$

Let $W(x, y, z,\lambda) = \frac{\lambda}{3}x+\frac{\lambda}{3}y+\frac{\lambda}{3}z$ for all x, y, z∈ X and λ∈ (0, 1]. Then (L, D, W) is a Quasi convex D-metric space.

Definition 6.3: A subset M of a Quasi convex D-metric space (X, D, W) is said to be a Quasi convex if W(x, y, z, λ)∈ M for all x, y, z∈ M and for all λ with 0<λ≤1.

Proposition 6.4: Every Quasi convex D-metric space (X, D, W) is a weak convex D-metric space.

Proposition 6.5: Let (X, D, W) be a Quasi convex D-metric space, u∈ X, r >0. Then

(i) If $\{K_\alpha:\alpha\in\Delta\}$ is a family of convex subsets of the Quasi convex D-metric space (X, D, W), then $\bigcap_{\alpha\in\Delta}K_\alpha$ is also a quasi convex subset in (X, D, W).

(ii) The balls $S_1(u, r)$.(in the sense of Naidu), $\hat{S}(u, r)$.(in the sense of Naidu) and $S_2(u, r)$ (in the sense of Asim) in (X, D, W) are quasi convex subsets of (X, D, W). The balls S*(u, r) and S*[u, r] are quasi convex in (X, D, W).

Definition 6.6: A Quasi convex D-metric space (X, D, W) is said to have the Property(QDC) if every bounded decreasing sequence of nonempty closed quasi convex subsets of (X, D, W) has nonempty intersection.

Proposition 6.7: Let (X, D, W) be a Quasi convex D-metric space. Let A⊆X. If (X, D, W) has the Property(QDC), then A_c is nonempty, closed and Quasi convex.

Proof: Analogous to Proposition 2.9.

Theorem 6.8:Let M be a nonempty compact subset of a Quasi convex D-metric space (X, D, W) and let K be the least closed weak convex set containing M. If the diameter δ(M) is positive, then there exists an element u∈ K such that sup{D(y, y, u): y∈ M}< δ(M).

Proof: Since (X, D, W) is a quasi convex D-metric space, by using Proposition 6.4, (X, D, W) is a weak convex D-metric space. Then K is also a weak convex set containing M in the weak convex D-metric space (X, D, W). By applying Proposition 5.8, there exists an element u∈ K such that sup{d(x, u):x∈ M}<δ(M).

Definition 6.9: A quasi convex D-metric space (X, D, W) is said to have normal structure if for each closed bounded quasi convex subset A of (X, D, W) which contains at least two points, there exists x∈ A which is not a diametral point of A.

Theorem 6.10: Let (X, D, W) be a quasi convex D-metric space. Suppose that (X, D, W) has the Property(QDC). Let K be a nonempty bounded closed weak convex subset of (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

Proof: Since (X, D, W) is a quasi convex D-metric space, by using Proposition 6.4, (X, D, W) is a weak convex D-metric space. Then K is also a bounded closed weak convex subset of weak convex D-metric space (X, D, W) with normal structure. If T is a nonexpansive mapping of K into itself, then by applying Theorem 5.9, T has a fixed point in K.

Definition 6.11: Let K be a compact Quasi convex D-metric space. Then a family $\mathcal{F}$ of nonexpansive mappings T of K into itself is said to have invariant property in K if for any compact and quasi convex subset A of K such that TA⊆A for each T∈ $\mathcal{F}$, there exists a compact subset M⊆A such that TM=M for each T∈ $\mathcal{F}$.

Theorem 6.12: Let (X, D, W) be a Quasi convex D-metric space. Let K be a compact Quasi convex D-metric space. If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then the family $\mathcal{F}$ has a common fixed point.

Proof: Since (X, D, W) is a quasi convex D-metric space, by using Proposition 6.4, (X, D, W) is a weak convex D-metric space. Then K is also a weak convex D-metric space (X, D, W). If $\mathcal{F}$ is a family of nonexpansive mappings with invariant property in K, then by applying Theorem 5.11 the family has a common fixed point in K.

7. Conclusion

Thus we have introduced and studied four types of convex structures in a D-metric space. We established some fixed point theorems in these structures. The link between the convex structures is given below:

(i) If (X, D, W) is a Strong convex D-metric space then (X, D, W) is a convex D-metric space.
(ii) If (X, D, W) is a J-convex D-metric space then (X, D, W) is a Strong convex D-metric space.
(iii) If (X, D, W) is a convex D-metric space then (X, D, W) is a weak convex D-metric space.
(iv) If (X, d, W) is a Quasi convex D-metric space then (X, D, W) is a Weak convex D-metric space.

8. References

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