On Hankel type transform of generalized Mathieu series

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Research Article

Abstract: In this paper, by using integral representations for several Mathieu type series, a number of integral transforms of Hankel type are derived here for general families of Mathieu type series. These results generalize the corresponding ones on the Fourier transforms of Mathieu type series, obtained recently by Elezovic et.al. [4], Tomovski [19] and Tomovski and Vu Kim Tuan [20].

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Introduction:

The infinite series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2}(r\in\mathbb{R}^+) \tag{1.1}$$

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [8] on elasticity of solid bodies. An integral representation of (1.1) is given by (See [5]) $S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t-1}\,dt$

(1.2)

Several interesting results dealing with integral representations and bounds for a slight generalization of the Mathieu series with a fractional power, defined as

$$S_{\alpha,\beta}(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^{\alpha-\beta}}\;\left(\left(r\in\mathbb{R}^+;\,(\alpha-\beta)>1\right)\right) \tag{1.3}$$

can be found in the works by Diananda [2]. Tomovski and Trencevski [16], Cerone and Lenard [1] and Tomovski [18]. Srivastava and Tomovski [14] defined a family of generalized Mathieu series

$$S_{\alpha,\beta}^{(p,q)}(r:a)=S^{(p,q)}\left(\left(r:\{\{a_k\}\}_{k=1}^{\infty}\right)\right)=\sum_{n=1}^{\infty}\frac{2a_n^q}{(\left(a_n^p+r^2\right))^{p-q}},\;\left(\left(r,p,q,((\alpha-\beta))\in\mathbb{R}^+\right)\right), \tag{1.4}$$

where it is tacitly assumed that the positive sequence

$$a=\{\{a_n\}\}_{n=1}^{\infty}=\{\{a_1,a_2,a_3,.......\}\}\;\left(\left(\lim_{n\to\infty}a_n=\infty\right)\right)$$

is so chosen that the infinite series in definition (1.4) converges, that is that the following auxiliary series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{((\alpha-\beta))p-q}}$$

is convergent. Comparing the definitions (1.1), (1.3) and (1.4), we see that

$$S_2(r) = S(r)\, and\, S_{\alpha,\beta}(r) = S_{\alpha,\beta}^{(2,1)}((r,\{\{k\}\})).$$

Furthermore, the special cases $S_2^{(2,1)}((r:\{\{a_k\}\})), S_{\alpha,\beta}^{(2,1)}((r:\{\{k^r\}\}))$ and $S_{\alpha,\beta}^{((p,p/2))}((r:\{\{k\}\}))$ have been investigated by Qi [13], Diananda [2], Tomovski [17] and Cerone-Lenard [1].

Definitions and formulas:

In this section we give some definitions and formulas needed for computation of the Hankel transforms of $S((r)), S_{\alpha,\beta,1}((r)), S_{\alpha,\beta}^{((p,q))}\left(r:\left\{r^{\frac{2}{p}}\right\}\right)$ *and* $S_{\alpha,\beta}^{((p,q))}((r:\{\{k^r\}\}))$.

In order to evaluate the Hankel type transform of $S((r))$, we first get $\alpha-\beta=\frac{1}{2}$ in Sonine-Schafheitlin formula {see example, [6, p.692]) :

$$\int_0^{\infty} t^{\lambda} J_{\alpha'-\beta'}(at)\, J_{\alpha-\beta}((bt))\, dt = \frac{a^{\alpha-\beta}\Gamma((\alpha'-\beta'+3\alpha+\beta))/2}{2^{\lambda}\, b^{3\alpha'+\beta'-\lambda}\, \Gamma((3\alpha'+\beta'))\, \Gamma((((\lambda-\alpha'+\beta'+3\alpha+\beta))/2))}$$
$$\times {}_2F_1\left(\frac{\alpha'-\beta'+3\alpha+\beta-\lambda}{2}, \frac{3\alpha'+\beta'-\alpha+\beta-\lambda}{2}; 3\alpha'+\beta'; \frac{a^2}{b^2}\right) \tag{2.1}$$
$$((R((\alpha'-\beta'+3\alpha+\beta)) > R((\lambda)) > -1;\ 0<a<b))$$

with a corresponding expression for the case when $0<b<a$, which is obtained from (2.1) by interchanging a and b, and also $((\alpha'-\beta'))\, and\, ((\alpha-\beta))$. In view of the relationship

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \tag{2.2}$$

we find from the Sonine-Schafheitlin formula (2.1) that

$$\int_0^{\infty} t^{-\lambda-\frac{1}{2}} J_{\alpha'-\beta'}((at))\sin((bt))\, dt = \sqrt{\frac{\pi b}{2}}\ \frac{a^{\alpha'-\beta'}\Gamma\left[\frac{1}{2}((2((2\alpha'+\beta'))-\lambda))\right]}{2^{\lambda} b^{3\alpha'+\beta'-\lambda}\Gamma((3\alpha'+\beta'))\Gamma\left[\left(\frac{1}{2}((\lambda+2((\alpha'+2\beta'))))\right)\right]}$$
$$\times {}_2F_1\left(\frac{2((2\alpha'+\beta'))-\lambda}{2}, \frac{2\alpha'-\lambda}{2}; 3\alpha'+\beta'; \frac{a^2}{b^2}\right)$$
$$((R((2((2\alpha'+\beta')))) > R((\lambda)) > -1;\ 0<a<b)), \tag{2.3}$$

together with the corresponding integral derived from the analogue of (2.1) for the case when $0<b<a$. Thus by further applying integral formula (2.3) and its companion when $0<b<a$, we obtain $((with\ \lambda=0,\ \alpha'-\beta'\to\alpha-\beta))$:

$$\int_0^\infty r^{-((\alpha+\beta))}\sin((rt))\,J_{\alpha-\beta}((rx))\,dr=\sqrt{\frac{\pi}{2}}\begin{cases}\Omega^{((1))}_{\alpha,\beta}((x;t)):0<x<t,\left(R((\alpha-\beta))>-\frac{1}{2}\right)\\ \Omega^{((2))}_{\alpha,\beta}((x;t)):0<t<x\end{cases}, \tag{2.4}$$

where

$$\Omega^{((1))}_{\alpha,\beta}((x;t))=\frac{x^{\alpha-\beta}}{t^{2\alpha}\,\Gamma((3\alpha+\beta))\,\Gamma((\alpha+2\beta))}\;2F_1\left(2\alpha+\beta,\alpha;3\alpha+\beta;\frac{x^2}{t^2}\right),\ ((0<x<t))$$

$$\Omega^{((2))}_{\alpha,\beta}((x;t))=\frac{t}{x^{3((\alpha+\beta))}}\frac{\Gamma((2\alpha+\beta))}{\Gamma\left(\frac{3}{2}\right)\Gamma((\alpha))}\;2F_1\left(\alpha+2\beta,\alpha+2\beta;\frac{3}{2};\frac{t^2}{x^2}\right),\ ((0<t<x)),\left(R((\alpha-\beta))>-\frac{1}{2}\right).$$

If we apply the Sonine-Schafheitlin formula (2.1), first with

$$\lambda=-\alpha'-3\beta',\ \alpha'-\beta'\to-2\beta',\ a=t,\ b=x,((0<a<b)),$$

and then with $\lambda=-\alpha'-3\beta',\ \alpha'-\beta'\to\alpha-\beta,\ \alpha-\beta\to-2\beta',\ a=x,b=t\,((0<b<a))$, we get

$$\int_0^\infty r^{\alpha'+3\beta'}J_{-2\beta'}((rt))\,J_{\alpha-\beta}((rx))\,dr=\begin{cases}\Theta^{((1))}_{\alpha,\beta}((\alpha'-\beta';x,t)):0<t<x\\ \Theta^{((2))}_{\alpha,\beta}((\alpha'-\beta':x,t)):0<x<t\end{cases},$$

where $\left(R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}\right).$ (2.5)

$$\Theta^{((1))}_{\alpha,\beta}((\alpha'-\beta':x,t))=\frac{t^{-2\beta'}\Gamma((2\alpha+\beta))}{2^{-\alpha'-3\beta'}\,x^{3((\alpha+\beta))}\,\Gamma((2\alpha'))\,\Gamma((2\alpha+\beta))}\;{}_2F_1\left(2\alpha+\beta,\ \alpha+2\beta:2\alpha':\frac{t^2}{x^2}\right)$$

$$\left(0<t<x,\ R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}\right). \tag{2.6}$$

$$\Theta^{((2))}_{\alpha,\beta}((\alpha'-\beta';\ x,t))=\frac{x^{\alpha-\beta}\,\Gamma((2\alpha+\beta))}{2^{\alpha'-3\beta'}\,t^{\alpha-\beta+3\alpha'+5\beta'}\,\Gamma((3\alpha'+\beta'))\,\Gamma((\alpha'-\beta'-\alpha))}$$

$$\times{}_2F_1\left(2\alpha+\beta,\ \frac{\alpha-\beta}{2}+\frac{3\alpha'+7\beta'}{2};3\alpha+\beta;\frac{x^2}{t^2}\right)$$

$$\left(0<x<t,\ R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}\right). \tag{2.7}$$

For the evaluation of the Hankel type transform of the general Mathieu type series $S^{((p,q))}_{\alpha',\beta'}\left(r;\left\{r^{\frac{2}{p}}\right\}\right)$, we need the definition of the Meijer G-function and the Fox H-function (for more details, see e.g. in [3, Vol. 1, [12]).

Definition 1: By a Fox's H-function we mean a generalized hypergeometric function defined by means of the Mellin-Barnes-type contour integral.

$$H_{u,v}^{m,n}\left[\delta\middle|_{b_v,q_v}^{a_u,p_u}\right]=\frac{1}{2\pi i}\int_l \frac{\prod_{k=1}^{m}\Gamma((b_k^{-}s\,q_k))\prod_{j=1}^{n}\Gamma((1-a_j+s\,p_j))}{\prod_{k=m+1}^{q}\Gamma((1-b_k+s\,q_k))\prod_{j=n+1}^{p}\Gamma((a_j-s\,p_j))} \tag{2.8}$$

where t is a suitable contour in $\mathbb{C}$, the orders $((m,n,u,v))$ are integers, $0\le m\le q$, $0\le n\le p$ and the parameters $a_j\in\mathbb{R}$, $p_j>0\ j=1,2,.....p, b_k\in\mathbb{R}, q_k>0,\ k=1,2,\ ....q$ are such that $p_j\,((b_k+l))\neq q_k\,((a_j-l'-1)),\ l,l'=0,1,2,\ \ldots\ldots..$

Many special functions are particular cases of the H-function.

For example, if $p_l=q_j=1((l=1,2,.....p;\,j=1,2,.....q))$, it reduces to the Meijer G-function $G_{u,v}^{m,n}((\delta))$ (for definition and properties, See [3, Vol.1]) :

$$\mathrm{H}_{u,v}^{m,n}\left[\delta\middle|_{((b_v,q_v))}^{((a_u,p_u))}\right]=G_{u,v}^{m,n}\left[\delta\middle|_{((b_v))}^{((a_u))}\right] \tag{2.9}$$

On the other hand, the Fox-Wright Ψ - function is the following special case of the H-function, see for example in [15]:

$${}_u\Psi_v\left[((a_u,p_u));((b_v,q_v));x\right]=H_{u,v+1}^{1,\mu}\left[-x\middle|_{((0,1)),\,((1-b_v,q_v))}^{((1-a_u,p_u))}\right] \tag{2.10}$$

Next we use of the following Miller transform of a product of two hypergeometric functions, proven by Miller and Srivastava [9] (see also [12, Sect. 2.22, p. 333]).

$$F((s))=\int_0^\infty r^{s-1}\,{}_0F_1((-;1+\alpha'-\beta';-a^2r^2))\,{}_1F_2((p;q,\,3\alpha+\beta;-b^2r^2))\,dr$$

$$=\frac{1}{2a^s}\frac{\Gamma((3\alpha'+\beta'))\Gamma((3\alpha+\beta))\Gamma((q))}{\Gamma((p))}G_{3,3}^{1,2}\left[\frac{b^2}{a^2}\middle|_{0,-\alpha+\beta,1-q}^{1-s/2,1-p,3\alpha'+\beta'-s/2}\right] \tag{2.11}$$

$$((a>0,b>0,0<R((s))<R((5\alpha'+3\beta'+\alpha-\beta+q-p))),0<R((s))<R((4\alpha'+2\beta'+2p)).$$

Substituting the relation (See [11, p. 727])

$${}_0F_1((-;3\alpha'+\beta';-a^2r^2))=\frac{\Gamma((3\alpha'+\beta'))\,J_{\alpha'-\beta'}((2ar))}{((ar))^{\alpha'-\beta'}} \tag{2.12}$$

into the integral formula (2.11) with

$$\alpha'-\beta'\to\alpha-\beta,\ s=4\alpha'+2\beta'\to4\alpha+3\beta,\ a\to\frac{x}{2},\ p\to\alpha'-\beta',\ q\to\alpha'-\beta'-\frac{q}{p},$$

$\alpha-\beta=\alpha'-\beta'-\frac{q}{p}-\frac{1}{2}$, $b=\frac{t}{2}$, we get

$$\int_0^\infty r^{\alpha+\beta}\,J_{\alpha-\beta}((rx))\,{}_1F_2\left(\alpha'-\beta';\,\alpha'-\beta'-\frac{q}{p},\,2\alpha'-\frac{q}{p};-\frac{r^2t^2}{4}\right)dr$$

$$=\frac{\sqrt{2}}{x^{\frac{3}{2}}}\,\frac{\Gamma\left(2\alpha'-\frac{q}{p}\right)\Gamma\left(\alpha'-\beta'-\frac{q}{p}\right)}{\Gamma((\alpha'-\beta'))}G_{3,3}^{1,2}\left[\frac{t^2}{x^2}\middle|_{0,\ \ 2\beta'+\frac{q}{p},\ \ \alpha'+3\beta'+\eta p}^{\beta',\ \ \alpha'+3\beta',\ \ \frac{3\alpha-\beta}{2}-\frac{\alpha'}{2}+\frac{\beta'}{2}}\right] \tag{2.13}$$

$$\left(R((\alpha-\beta))>-\frac{3}{2}, R((\alpha'-\beta'))>0,\ R\left(\alpha'-\beta'-2\,\frac{q}{p}\right)>0,\ t>0,\ x>0\right).$$

In order to evaluate the Hankel type transform $S^{((p,q))}_{\alpha',\beta'}\left(\left(r;\left\{\left\{k^r\right\}\right\}\right)\right)$, we apply the known integral formula from [11, p.355] with $p=1,\ q=2,\ r=2,\ p=\frac{3}{2}$:

$$\int_0^\infty x^{\alpha+\beta} J_{\alpha-\beta}((\sigma x))\, H^{1,1}_{1,2}\left[\omega x^2 \middle|{}^{((\alpha'+3\beta',1))}_{((0,1))\,((1-r\,((\alpha'-\beta'))\,p-q)),\,rp}\right]dx = \frac{\sqrt{2}}{\sigma^{\frac{3}{2}}} H^{1,2}_{3,2}\left[\frac{4\omega}{\sigma^2}\middle|{}^{((\beta,1))\ ((\alpha'+3\beta',1)),\ ((\alpha,1))}_{((0,1)),\ ((1-\gamma((\alpha'-\beta'))\,p-q)),\ \gamma p}\right]$$

(2.14)

$$\left(\omega\in\mathbb{R},\ \sigma\in\mathbb{R}^+,\ R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}\right).$$

Using relation (2.10) with $p=1,\ q=1,$ by (2.14) we get the following integral formula:

$$\int_0^\infty x^{\alpha+\beta} J_{\alpha-\beta}((\sigma x))\ {}_1\Psi_1\left[((\alpha'-\beta',1));\left(\gamma(((\alpha'-\beta))\,p-q)),\gamma p\right));-x^2\,t^{\gamma p}\right]dx$$

$$=\frac{\sqrt{2}}{\sigma^{\frac{3}{2}}} H^{1,2}_{3,2}\left[\frac{4t^{\gamma p}}{\sigma^2}\middle|{}^{((\beta,1)),((\alpha'+3\beta',1)),((\alpha,1))}_{((0,1)),((1-\gamma((\alpha'-\beta'))\,p-q,\gamma p))}\right] \tag{2.15}$$

$$\left(t\in\mathbb{R}^+,\sigma\in\mathbb{R}^+,\ R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}\right).$$

3. Evaluation of Hankel type transforms:

The Hankel type transform of order $\alpha-\beta$ is defined by

$$H_{\alpha,\beta}((f((r))))((x))=\int_0^\infty f((r))\,J_{\alpha-\beta}((rx))\ ((rx))^{\alpha+\beta}\,dx \tag{3.1}$$

where $J_{\alpha-\beta}$ is the Bessel function of the first kind and of order $((\alpha-\beta))$ with $R((\alpha-\beta))\geq-\frac{1}{2}$.

In view of the relationship (2.1) and $J_{-\frac{1}{2}}((x))=\sqrt{\frac{2}{\pi x}}\ \cos x$. (3.2)

The Hankel type transform reduces to the Sin-Fourier and Cos-Fourier transforms:

$$((\Im_s f))((x))=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin((rx))\ f((r))\ dr,$$

$$((\Im_c f))((x))=\sqrt{\frac{2}{\pi}}\int_0^\infty \cos((rx))\ f((r))\ dr.$$

For a generalization of the Hankel type transform as well as the Sin-Fourier and Cos-Fourier transforms, see Luchko and Kiryakova [7].

Using the integral formula (2.4), we first evaluate the Hankel type transform of $S((r))$:

$$H_{\alpha,\beta}((S((r))))((x))=\int_0^\infty S((r))\,J_{\alpha-\beta}((rx))\ \ ((rx))^{\alpha+\beta}\,dr$$

$$=\sqrt{x}\int_0^\infty \frac{1}{\sqrt{r}}\left(\int_0^\infty \frac{t\sin 9\left(rt\right))}{e^t-1}\right)J_{\alpha-\beta}\left(\left(rx\right)\right)dr$$

$$=\sqrt{x}\int_0^\infty \frac{t}{e^t-1}\left(\int_0^\infty r^{-((\alpha-\beta))}\sin(\left(rt\right))\,J_{\alpha-\beta}\left(\left(rx\right)\right)dr\right)dt$$

$$=\sqrt{\frac{\pi x}{2}}\left(\int_0^x \frac{t}{e^t-1}\Omega_{\alpha,\beta}^{((2))}\left(\left(x;t\right)\right)\right)dt+\int_x^\infty \frac{t}{e^t-1}\Omega_{\alpha,\beta}^{((1))}\left(\left(\left(\left(x;t\right)\right)dt\right)\right). \qquad (3.3)$$

Using the relation [6, p.1041]

$$2F_1\left(1,\frac{1}{2};\frac{3}{2}:z^2\right)=\frac{1}{2z}\,ln\,\left(\frac{1+z}{1-z}\right),$$

we get

$$\Omega_{\frac{1}{2}}^{((1))}\left(\left(x;t\right)\right)=\frac{\sqrt{x}}{t\,\Gamma((3/2))\Gamma((1/2))}\,2F_1\left(1,\frac{1}{2};\frac{3}{2};\frac{x^2}{t^2}\right)=\frac{1}{\pi\sqrt{x}}ln\left(\frac{t+x}{t-x}\right)\left(\left(0<x<t\right)\right),$$

$$\Omega_{\frac{1}{2}}^{((2))}\left(\left(x;t\right)\right)=\frac{t}{x\sqrt{x}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}2F_1\left(1,\frac{1}{2};\frac{3}{2};\frac{t^2}{x^2}\right)=\frac{1}{\pi\sqrt{x}}ln\left(\frac{x+t}{x-t}\right),\ \left(\left(0<t<x\right)\right).$$

Hence

$$H_{\frac{1}{2}}\left(\left(S\left(\left(r\right)\right)\right)\right)\left(\left(x\right)\right)=\left(\left(\Im_s\,S\left(\left(r\right)\right)\right)\right)\left(\left(x\right)\right)$$

$$=\sqrt{\frac{\pi x}{2}}\left(\int_0^x \frac{t}{e^t-1}\frac{1}{\pi\sqrt{x}}\,ln\,\left(\frac{x+t}{x-t}\right)dt+\int_x^\infty \frac{t}{e^t-1}\,\frac{1}{\pi\sqrt{x}}\,ln\left(\frac{t+x}{t-x}\right)dt\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left(\int_0^x \frac{t}{e^t-1}ln\left(\frac{x+t}{x-t}\right)dt+\int_x^\infty \frac{t}{e^t-1}ln\left(\frac{t+x}{t-x}\right)u\right)$$

$$=\frac{1}{\sqrt{2\pi}}PV\left(\int_0^\infty \frac{t}{e^t-1}ln\left(\left|\frac{t+x}{t-x}\right|\right)dt\right)\left(\left(x>0\right)\right), \qquad (3.4)$$

where the Cauchy Principal Value (PV) of the last integral is assumed to exist. By a direct computation, the same formula (3.4) was recently proved by Elezovic et. al [4].

The integral representation of $S_{3\alpha'+\beta'}\left(\left(r\right)\right)$, obtained by Cerone and Lenard in [1] is given by

$$S_{3\alpha'+\beta'}\left(\left(r\right)\right)=\frac{\sqrt{\pi}}{((2r))^{-2\beta'}\,\Gamma(\left(3\alpha'+\beta'\right))}\int_0^\infty \frac{t^{2\alpha'}}{e^t-1}J_{-2\beta'}\left(\left(rt\right)\right)\,dt\ \left(\left(r,\,\alpha'-\beta'\in\mathbb{R}^+\right)\right). \qquad (3.5)$$

Applying integral formula (2.5), we obtain

$$H_{\alpha,\beta}\left(\left(S_{3\alpha'+\beta'}\left(\left(r\right)\right)\right)\right)\left(\left(x\right)\right)=\int_0^\infty S_{3\alpha'+\beta'},\left(\left(r\right)\right)J_{\alpha-\beta}\left(\left(rx\right)\right)\left(\left(rx\right)\right)^{\alpha+\beta}\,dr$$

$$=\frac{\sqrt{\pi x}}{2^{-2\beta'}\,\Gamma(\left(3\alpha'+\beta'\right))}\int_0^\infty r^{\alpha'+3\beta'}\left(\int_0^\infty \frac{t^{2\alpha'}}{e^t-1}J_{-2\beta'}\left(\left(rt\right)\right)dt\right)J_{\alpha-\beta}\left(\left(rx\right)\right)dr$$

$$=\frac{\sqrt{\pi x}}{2^{-2\beta'}\,\Gamma(\left(3\alpha'+\beta'\right))}\int_0^\infty \frac{t^{2\alpha'}}{e^t-1}\left(\int_0^\infty r^{\alpha'+3\beta'}J_{-2\beta'}(\left(rt\right))\ J_{\alpha-\beta}\left(\left(rx\right)\right)\,dr\right)dt$$

$$=\frac{\sqrt{\pi x}}{2^{-2\beta'}\Gamma((3\alpha'+\beta'))}\left(\int_0^x \frac{t^{2\alpha'}}{e^t-1}\Theta_{\alpha,\beta}^{((1))}((\alpha'-\beta';\,x,t))\right)dt$$

$$+\int_x^\infty \frac{t^{2\alpha'}}{e^t-1}\,\Theta_{\alpha,\beta}^{((2))}((\alpha'-\beta';\,x,t))\,dt\,, \tag{3.6}$$

$$\left(\alpha'-\beta'\in\mathbb{R}^+,\ R((\alpha-\beta))>-\frac{1}{2},\ x>0\right).$$

In order to evaluate the Hankel type transform of $S_{\alpha',\beta'}^{((p,q))}\left(r;\left\{k^{\frac{2}{p}}\right\}\right)$, we apply its integral representation from [14]:

$$S_{\alpha',\beta'}^{((p,q))}\left(r;\left\{k^{\frac{2}{p}}\right\}\right)=\frac{2}{\Gamma\left(2\left[\alpha'-\beta'-\frac{q}{p}\right]\right)}\int_0^\infty \frac{x^{2\left(\alpha'-\beta'-\frac{q}{p}\right)-1}}{e^x-1}$$

$$\times\,{}_1F_2\left(\alpha'-\beta';\,\alpha'-\beta'-\frac{q}{p},\,2\alpha-\frac{q}{p};\frac{-r^2x^2}{4}\right)dx \tag{3.7}$$

$\left(r,\,p,\,q\in\ \mathbb{R}^+,\ \alpha'-\beta'-\frac{q}{p}>\frac{1}{2}\right)$,and integral formula (2.13). Thus we have

$$H_{\alpha,\beta}\left(S_{\alpha',\beta'}^{((p,q))}\left(r;\left\{r^{\frac{2}{p}}\right\}\right)\right)((x))=\int_0^\infty S_{\alpha',\beta'}^{((p,q))}\left(r;\left\{r^{\frac{2}{p}}\right\}\right)J_{\alpha-\beta}((rx))((rx))^{\alpha+\beta}\,dr$$

$$=\frac{2\sqrt{x}}{\Gamma\left(2\left[\alpha'-\beta'-\frac{q}{p}\right]\right)}\int_0^\infty \frac{t^2(\alpha'-\beta'-\frac{q}{p})-1}{e^t-1}$$

$$\times\left(\int_0^\infty r^{\alpha+\beta}\ 1F_2\left(\alpha'-\beta';\alpha'-\beta'-\frac{q}{p},\,2\alpha'-\frac{q}{p};-\frac{r^2t^2}{4}\right)J_{\alpha-\beta}((rx))\,dr\right)dt$$

$$=\frac{2\sqrt{2\pi}}{2^{2\left(\alpha'-\beta'-\frac{q}{p}\right)-1}x\Gamma((\alpha'-\beta'))}\left(\int_0^\infty \frac{t^{2\left(\alpha'-\beta'-\frac{q}{p}\right)-1}}{e^t-1}G_{3,3}^{1,2}\left(\frac{t^2}{x^2}\middle|\begin{matrix}\beta',\,\alpha'+3\beta',\,\frac{3\alpha-\beta}{2}+\frac{\alpha'-\beta'}{2}\\ 0,\,2\beta'+\frac{q}{p},\ \alpha'+3\beta'+\frac{q}{p}\end{matrix}\right)dt\right)$$

$$\left(R((\alpha-\beta))>-\frac{1}{2},\ \alpha'-\beta'-2\frac{q}{p}>0,\ \alpha'-\beta'-\frac{q}{p}>\frac{1}{2},x>0\right). \tag{3.8}$$

The integral representation of $S_{\alpha',\beta'}^{((p,q))}\left(\left(r;\{\{k^\gamma\}\}_{k=1}^\infty\right)\right)$ obtained by Srivastava and Tomovski [14] is given by

$$S_{\alpha,\beta}^{((p,q))}\left(\left(r;\{\{k^\gamma\}\}_{k=1}^\infty\right)\right)=\frac{2}{\Gamma((\alpha'-\beta'))}\int_0^\infty \frac{x^{\gamma(((\alpha'-\beta'))p-q))-1}}{e^x-1}\,{}_1\Psi_1\left[((\alpha'-\beta',1));\,((\gamma((\alpha'-\beta'))p-q)),\,\gamma p;-r^2x^{\gamma p}\right]dx$$

(3.9)

$((r, p, q, y \in \mathbb{R}^+,\ \gamma((((\alpha' - \beta'))p - q)) > 1))$.

Using this integral representation and (2.15), we get

$$H_{\alpha,\beta}((S_{\alpha',\beta'}^{((p,q))}((r;\{\{k^\gamma\}\})))((x))$$

$$= \int_0^\infty S_{\alpha',\beta'}^{((p,q))}((\gamma;\{\{k^\gamma\}\}))\, J_{\alpha-\beta}((yx))((rx))^{\alpha+\beta}\, dr$$

$$= \frac{2\sqrt{x}}{\Gamma(\alpha'-\beta')}\int_0^\infty \frac{t^{y(\alpha'-\beta')(p-q)-1}}{e^t-1}\left(\int_0^\infty r^{\alpha+\beta} J_{\alpha-\beta}((rx))\Psi_1\left[((\alpha'-\beta',1));((\gamma((\alpha'-\beta'))p-q)),\gamma p));-r^2t^{\gamma p}\right]dr\right)dt$$

$$= \frac{2\sqrt{2}}{x\Gamma((\alpha'-\beta'))}\int_0^\infty \frac{t^{\gamma((((\alpha'-\beta'))p-q))-1}}{e^t-1} H_{3,2}^{1,2}\left(\frac{4t^{\gamma p}}{x^2}\ \begin{matrix}((\beta,1)),((((\alpha'+3\beta')),1)),((\alpha,1)) \\ ((0,1)),((1-\gamma))((((\alpha'-\beta'))p-q)),\gamma p\end{matrix}\right) dt\ ,$$

$$\left(R((\alpha'-\beta'))>0,\ R((\alpha-\beta))>-\frac{3}{2}, \gamma((((\alpha'-\beta'))p-q))>1, x>0\right) \qquad (3.10)$$

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