Common Fixed Point Theorems for Sequence of Mappings in $\mathcal{M}$ - Fuzzy Metric Space

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Research Article

Abstract: In this paper we prove some common fixed point theorems for sequence of mappings in complete $\mathcal{M}$ – fuzzy metric space.

Keywords: Complete $\mathcal{M}$ – Fuzzy metric space, Common fixed point, Sequence of maps.

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1. Introduction and Preliminaries

Zadeh [16] introduced the concept of fuzzy sets in 1965. George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] and defined the Hausdorff topology of fuzzy metric spaces. Many authors [4, 7] have proved fixed point theorems in fuzzy metric space. Recently Sedghi and Shobe [13] introduced D^*-metric space as a probable modification of the definition of D-metric introduced by Dhage [1], and prove some basic properties in D^*-metric spaces. Using D^*-metric concepts, Sedghi and Shobe define $\mathcal{M}$ – fuzzy metric space and proved a common fixed point theorem in it. In this paper we prove some common fixed point theorems for sequence of mappings in complete $\mathcal{M}$ – fuzzy metric space.

Definition 1.1: Let X be a nonempty set. A generalized metric (or D^* - metric) on X is a function: $D^* : X^3 \to [0, \infty)$, that satisfies the following conditions for each $x, y, z, a \in X$

(i) $D^*(x, y, z) \geq 0$,

(ii) $D^*(x, y, z) = 0$ iff $x = y = z$,

(iii) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,

(iv) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*), is called a generalized metric (or D^* - metric) space.

Example 1.2: Examples of D^* - metric are

(a) $D^*(x, y, z) = \max\{ d(x, y), d(y, z), d(z, x) \}$,

(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

Definition 1.3: A fuzzy set $\mathcal{M}$ in an arbitrary set X is a function with domain X and values in [0, 1].

Definition 1.4: A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) $*$ is associative and commutative,

(ii) $*$ is continuous,

(iii) $a * 1 = a$ for all $a \in [0, 1]$,

(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Examples for continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 1.5: A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$ – fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

(FM – 1) $\mathcal{M}(x, y, z, t) > 0$

(FM – 2) $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$

(FM – 3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function

(FM – 4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$

(FM – 5) $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous

(FM – 6) $\lim_{t \to \infty} \mathcal{M}(x, y, z, t) = 1$.

Example 1.6: Let X be a nonempty set and D^* is the D^* - metric on X. Denote $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$ – fuzzy metric space.

Example 1.7: Let $(X, M, *)$ be a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \to [0, 1]$ by

$\mathcal{M}(x, y, z, t) = M(x, y, t)*M(y, z, t)*M(z, x, t)$

for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$ – fuzzy metric space.

Lemma 1.8:([13]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Lemma 1.9:([13]) Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing with respect to t, for all x, y, z in X.

Definition 1.10: Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{ y \in X: \mathcal{M}(x, y, y, t) > 1 - r \}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition 1.11: Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to be converges to a point $x \in X$ if $lim_{n \to \infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$

(b) $\{x_n\}$ is called Cauchy sequence if $lim_{n \to \infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$

(c) A $\mathcal{M}$ – fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark 1.12: Since * is continuous, it follows from (*FM-4*) that the limit of the sequence is uniquely determined.

Definition 1.13: Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space, then $\mathcal{M}$ is called of first type if for every $x, y \in X$ we have

$$\mathcal{M}(x, x, y, t) \geq \mathcal{M}(x, y, z, t)$$

for every $z \in X$. Also it is called of second type if for every $x, y, z \in X$ we have

$\mathcal{M}(x, y, z, t) = M(x, y, t)*M(y, z, t)*M(z, x, t)$.

Example 1.14: Let X be a nonempty set and D^* is the D^* - metric on X. If we define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)},$$

where $D^*(x, y, z) = d(x, y) + d(y, z) + d(x, z)$, then $\mathcal{M}$ is first type.

Example 1.15: If $(X, M, *)$ is a fuzzy metric and $M(x, y, t) = \dfrac{t}{t + d(x, y)}$, then

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + d(x, y)} * \frac{t}{t + d(y, z)} * \frac{t}{t + d(x, z)}$$

is second type.

Definition 1.16: A point $x \in X$ is said to be a fixed point of the map $T: X \to X$ if $Tx = x$.

Definition 1.17: A point $x \in X$ is said to be a common fixed point of sequence of maps $T_n: X \to X$ if $T_n(x) = x$ for all n.

2. Main Results

Theorem 2.1: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$$3\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, T_i x, t/k) + \mathcal{M}(y, y, T_j y, t/k)\}$$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$$\begin{aligned}
3\mathcal{M}&(x_{n+1}, x_{n+2}, x_{n+2}, t) \\
&= 3\mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t) \\
&\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x_n, x_n, T_{n+1}x_n, t/k) \\
&\qquad + \mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k)\} \\
&= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x_n, x_n, x_{n+1}, t/k) \\
&\qquad + \mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k)\} \\
&= 2\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k) \\
&\geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t)
\end{aligned}$$

Therefore,

$$2\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$$

That is, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$.

Continuing this way we get

$$\begin{aligned}
\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) &\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) \\
&\geq \mathcal{M}(x_{n-1}, x_n, x_n, t/k^2) \\
&\quad \vdots \\
&\geq \mathcal{M}(x_0, x_1, x_1, t/k^{n+1}).
\end{aligned}$$

Now for any positive integer p and $t > 0$, we have

$$\begin{aligned}
\mathcal{M}&(x_n, x_{n+p}, x_{n+p}, t) \\
&\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overset{p\ times}{*\ldots\ldots*} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p}) \\
&\geq \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^n}) \overset{p\ times}{*\ldots\ldots*} \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^{n+p-1}})
\end{aligned}$$

Therefore, by (*FM-6*), we have

$$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overset{p\ times}{*\ldots\ldots*} 1 = 1$$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a fixed point of $\{T_n\}$ for all n.

Now we have

$$\begin{aligned}
3\mathcal{M}(T_m x, x, x, t) &= lim_{n \to \infty}\ 3\mathcal{M}(T_m x, x_{n+2}, x_{n+2}, t) \\
&= lim_{n \to \infty}\ 3\mathcal{M}(T_m x, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t) \\
&\geq lim_{n \to \infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x, x, T_m x, t/k) \\
&\qquad + \mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k)\}
\end{aligned}$$

$= lim_{n \to \infty}$ $\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k) + \mathcal{M}(x, x, T_m x, t/k) + \mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k)\}$

$= \{\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, x, T_m x, t/k) + \mathcal{M}(x, x, x, t/k)\}$

$= \{1 + \mathcal{M}(x, x, T_m x, t/k) + 1\}$

$= 2 + \mathcal{M}(x, x, T_m x, t/k)$

$\geq 2 + \mathcal{M}(x, x, T_m x, t)$

Therefore, $2\mathcal{M}(T_m x, x, x, t) \geq 2$

That is, $\mathcal{M}(T_m x, x, x, t) \geq 1$

Hence $\mathcal{M}(T_m x, x, x, t) = 1$, for all $t > 0$.

Therefore, $T_m x = x$.

Hence $T_n x = x$ for all n.

Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_n y = y$ for all n. Then

$3\mathcal{M}(x, y, y, t) = 3\mathcal{M}(T_i x, T_j y, T_j y, t)$

$\geq \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, T_i x, t/k) + \mathcal{M}(y, y, T_j y, t/k)\}$

$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, x, t/k) + \mathcal{M}(y, y, y, t/k)\}$

$= \mathcal{M}(x, y, y, t/k) + 2$

$\geq \mathcal{M}(x, y, y, t) + 2$

Therefore, $2\mathcal{M}(x, y, y, t) \geq 2$

That is, $\mathcal{M}(x, y, y, t) \geq 1$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.

Therefore, $x = y$.

which is contradiction to $x \neq y$.

Hence $\{T_n\}$ have a unique common fixed point.

This completes the proof.

Remark 2.2: From the above theorem we have,

$\mathcal{M}(T_i x, T_j y, T_j y, t)$

$\geq \frac{1}{3}\{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, T_i x, t/k) + \mathcal{M}(y, y, T_j y, t/k)\}$

$\geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, T_i x, t/k), \mathcal{M}(y, y, T_j y, t/k)\}$

Therefore, $\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, T_i x, t/k), \mathcal{M}(y, y, T_j y, t/k)\}$.

Hence we get the following corollary.

Corollary 2.3: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, T_i x, t/k), \mathcal{M}(y, y, T_j y, t/k)\}$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Remark 2.4: By taking $T_i = T_j = T$ in the above corollary, we get the following corollary 2.5.

Corollary 2.5: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(Tx, Ty, Ty, t) \geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, Tx, t/k), \mathcal{M}(y, y, Ty, t/k)\}$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 2.6: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, x, T_i x, t/k)*\mathcal{M}(y, y, T_j y, t/k)\}$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) = \mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_n, x_n, T_{n+1}x_n, t/k) *\mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k)\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_n, x_n, x_{n+1}, t/k) *\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k)\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) *\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$, which implies that

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$.

Continuing this way we get

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$

$\geq \mathcal{M}(x_{n-1}, x_n, x_n, t/k^2)$

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$\geq \mathcal{M}(x_0, x_1, x_1, t/k^{n+1})$.

Now for any positive integer p and $t > 0$, we have

$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overbrace{*\ldots\ldots*}^{p\ \ times} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})$

$\geq \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^n}) \overbrace{*\ldots\ldots*}^{p\ \ times} \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^{n+p-1}})$

Therefore, by $(FM\text{-}6)$, we have

$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overbrace{*\ldots\ldots*}^{p\ \ times} 1 = 1$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.
Now we prove that x is a fixed point of $\{T_n\}$ for all n.
Now we have

$$\begin{aligned}
\mathcal{M}(T_m x, x, x, t) &= lim_{n\to\infty}\ \mathcal{M}(T_m x, x_{n+2}, x_{n+2}, t)\\
&= lim_{n\to\infty}\ \mathcal{M}(T_m x, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)\\
&\geq lim_{n\to\infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x, x, T_m x, t/k)\\
&\qquad *\mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k)\}\\
&= lim_{n\to\infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x, x, T_m x, t/k)\\
&\qquad *\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k)\}\\
&= \{\mathcal{M}(x, x, x, t/k)*\mathcal{M}(x, x, T_m x, t/k)*\mathcal{M}(x, x, x, t/k)\}\\
&= \{1*\mathcal{M}(x, x, T_m x, t/k)*1\}\\
&= \mathcal{M}(x, x, T_m x, t/k)\\
&= \mathcal{M}(T_m x, x, x, t/k)\\
&\ \ \vdots\\
&\geq \mathcal{M}(T_m x, x, x, t/k^n)\\
&\to 1 \text{ as } n\to\infty
\end{aligned}$$

Hence $\mathcal{M}(T_m x, x, x, t) = 1$, for all $t > 0$.
Therefore, $T_m x = x$.
Hence $T_n x = x$ for all n.
Therefore x is a common fixed point of $\{T_n\}$.
Uniqueness: Suppose $x \neq y$ such that $T_n y = y$ for all n.
Then

$$\begin{aligned}
\mathcal{M}(x, y, y, t) &= \mathcal{M}(T_i x, T_j y, T_j y, t)\\
&\geq \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, x, T_i x, t/k)\\
&\qquad *\mathcal{M}(y, y, T_j y, t/k)\}\\
&= \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, x, x, t/k)*\mathcal{M}(y, y, y, t/k)\}\\
&= \{\mathcal{M}(x, y, y, t/k)*1*1\}\\
&= \mathcal{M}(x, y, y, t/k)\\
&\ \ \vdots\\
&\geq \mathcal{M}(x, y, y, t/k^n)\\
&\to 1 \text{ as } n\to\infty
\end{aligned}$$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.
Therefore, $x = y$.
which is contradiction to $x \neq y$.
Hence $\{T_n\}$ have a unique common fixed point.
This completes the proof.

Remark 2.7: By taking $T_i = T_j = T$ in the above theorem, we get the following corollary 2.8

Corollary 2.8: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$$\mathcal{M}(Tx, Ty, Ty, t) \geq \{\mathcal{M}(x, y, y, t/k) * \mathcal{M}(x, x, Tx, t/k) * \mathcal{M}(y, y, Ty, t/k)\}$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 2.9: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$$\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, T_i x, t/k), \mathcal{M}(y, y, T_j y, t/k), \mathcal{M}(x, x, T_j y, 2t/k)\}$$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2, \ldots$
Now we prove that $\{x_n\}$ is a Cauchy sequence in X.
For $n \geq 0$, we have

$$\begin{aligned}
\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) &= \mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)\\
&\geq \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_n, x_n, T_{n+1}x_n, t/k),\\
&\qquad \mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k), \mathcal{M}(x_n, x_n, T_{n+2}x_{n+1}, 2t/k)\}\\
&= \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_n, x_n, x_{n+1}, t/k),\\
&\qquad \mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k), \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}\\
&= \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k),\\
&\qquad \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k), \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}\\
&= \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k),\\
&\qquad \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}\\
&\geq \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k),\\
&\qquad \mathcal{M}(x_n, x_n, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}\\
&= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}
\end{aligned}$$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$, which implies that

$$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k).$$

Continuing this way we get

$$\begin{aligned}
\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) &\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)\\
&\geq \mathcal{M}(x_{n-1}, x_n, x_n, t/k^2)\\
&\ \ \vdots\\
&\geq \mathcal{M}(x_0, x_1, x_1, t/k^{n+1}).
\end{aligned}$$

Now for any positive integer p and $t > 0$, we have

$$\begin{aligned}
\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) &\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overset{p\ times}{*\ldots\ldots*} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})\\
&\geq \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^n}) \overset{p\ times}{*\ldots\ldots*} \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^{n+p-1}})
\end{aligned}$$

Therefore, by (*FM-6*), we have

$$lim_{n\to\infty}\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overset{p\ times}{*\ldots\ldots*} 1 = 1$$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.
Now we prove that x is a fixed point of $\{T_n\}$ for all n.
Now we have

$\mathcal{M}(T_m x, x, x, t) = lim_{n \to \infty}\ \mathcal{M}(T_m x, x_{n+2}, x_{n+2}, t)$

$= lim_{n \to \infty}\ \mathcal{M}(T_m x, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$

$\geq lim_{n \to \infty}$ min $\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x, x, T_m x, t/k), \mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k), \mathcal{M}(x, x, T_{n+2}x_{n+1}, 2t/k)\}$

$= lim_{n \to \infty}$ min $\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x, x, T_m x, t/k), \mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k), \mathcal{M}(x, x, x_{n+2}, 2t/k)\}$

$=$ min $\{\mathcal{M}(x, x, x, t/k), \mathcal{M}(x, x, T_m x, t/k), \mathcal{M}(x, x, x, t/k), \mathcal{M}(x, x, x, 2t/k)\}$

$=$ min $\{1, \mathcal{M}(x, x, T_m x, t/k), 1, 1\}$

$= \mathcal{M}(x, x, T_m x, t/k)$

$= \mathcal{M}(T_m x, x, x, t/k)$

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$\geq \mathcal{M}(T_m x, x, x, t/k^n)$

$\to 1$ as $n \to \infty$

Hence $\mathcal{M}(T_m x, x, x, t) = 1$, for all $t > 0$.

Therefore, $T_m x = x$.

Hence $T_n x = x$ for all n.

Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_n y = y$ for all n. Then

$\mathcal{M}(x, y, y, t) = \mathcal{M}(T_i x, T_j y, T_j y, t)$

$\geq$ min $\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, T_i x, t/k),$
$\mathcal{M}(y, y, T_j y, t/k), \mathcal{M}(x, x, T_j y, 2t/k)\}$

$=$ min $\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, x, t/k),$
$\mathcal{M}(y, y, y, t/k), \mathcal{M}(x, x, y, 2t/k)\}$

$=$ min $\{\mathcal{M}(x, y, y, t/k), 1, 1, \mathcal{M}(x, y, y, 2t/k)\}$

$= \mathcal{M}(x, y, y, t/k)$

.

.

.

$\geq \mathcal{M}(x, y, y, t/k^n)$

$\to 1$ as $n \to \infty$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.

Therefore, $x = y$.

which is contradiction to $x \neq y$.

Hence $\{T_n\}$ have a unique common fixed point.

This completes the proof.

Remark 2.10: By taking $T_i = T_j = T$ in the above theorem, we get the following corollary 2.11

Corollary 2.11: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm * is defined by $a*b = \min\{a, b\}$ and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(Tx, Ty, Ty, t) \geq$ min $\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, Tx, t/k), \mathcal{M}(y, y, Ty, t/k), \mathcal{M}(x, x, Ty, 2t/k)\}$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 2.12: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm * is defined by $a*b = \min\{a, b\}$ and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_i x, T_j y, T_j y, t) \geq \{\mathcal{M}(x, y, y, t/k) * \mathcal{M}(x, x, T_i x, t/k) * \mathcal{M}(y, y, T_j y, t/k) * \mathcal{M}(x, x, T_j y, 2t/k)\}$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) = \mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_n, x_n, T_{n+1}x_n, t/k) *$
$\mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k) * \mathcal{M}(x_n, x_n, T_{n+2}x_{n+1}, 2t/k)\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_n, x_n, x_{n+1}, t/k)$
$* \mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k) * \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$
$* \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k) * \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)$
$* \mathcal{M}(x_n, x_n, x_{n+2}, 2t/k)\}$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)$
$* \mathcal{M}(x_n, x_n, x_{n+1}, t/k) * \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k) * \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$, which implies that

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$.

Continuing this way we get

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$

$\geq \mathcal{M}(x_{n-1}, x_n, x_n, t/k^2)$

.

.

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$\geq \mathcal{M}(x_0, x_1, x_1, t/k^{n+1})$.

Now for any positive integer p and $t > 0$, we have

$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$$\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overset{p\ \ times}{*\ldots\ldots*} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})$$

$$\geq \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^n}) \overset{p\ \ times}{*\ldots\ldots*} \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^{n+p-1}})$$

Therefore, by (*FM-6*), we have

$$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overset{p\ \ times}{*\ldots\ldots*} 1 = 1$$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a fixed point of $\{T_n\}$ for all n.

Now we have

$\mathcal{M}(T_m x, x, x, t) = lim_{n \to \infty}\ \mathcal{M}(T_m x, x_{n+2}, x_{n+2}, t)$

$= lim_{n\to\infty}$ $\mathcal{M}(T_mx, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$
$\geq lim_{n\to\infty}$ $\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x, x, T_mx, t/k)$
$*\mathcal{M}(x_{n+1}, x_{n+1}, T_{n+2}x_{n+1}, t/k)*\mathcal{M}(x, x, T_{n+2}x_{n+1}, 2t/k)\}$
$= lim_{n\to\infty}$ $\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x, x, T_mx, t/k)$
$*\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t/k)*\mathcal{M}(x, x, x_{n+2}, 2t/k)\}$
$= \{\mathcal{M}(x, x, x, t/k)*\mathcal{M}(x, x, T_mx, t/k)*\mathcal{M}(x, x, x, t/k)$
$*\mathcal{M}(x, x, x, 2t/k)\}$
$= \{1*\mathcal{M}(x, x, T_mx, t/k)*1*1\}$
$= \mathcal{M}(x, x, T_mx, t/k)$
$= \mathcal{M}(T_mx, x, x, t/k)$
.
.
.
$\geq \mathcal{M}(T_mx, x, x, t/k^n)$
$\to 1$ as $n \to \infty$

Hence $\mathcal{M}(T_mx, x, x, t) = 1$, for all $t > 0$.
Therefore, $T_mx = x$.
Hence $T_nx = x$ for all n.
Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_ny = y$ for all n. Then

$\mathcal{M}(x, y, y, t) = \mathcal{M}(T_ix, T_jy, T_jy, t)$
$\geq \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, x, T_ix, t/k)*\mathcal{M}(y, y, T_jy, t/k)$
$*\mathcal{M}(x, x, T_jy, 2t/k)\}$
$= \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, x, x, t/k)*\mathcal{M}(y, y, y, t/k)$
$*\mathcal{M}(x, x, y, 2t/k)\}$
$= \{\mathcal{M}(x, y, y, t/k)*1*1*\mathcal{M}(x, y, y, 2t/k)\}$
$= \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, y, y, 2t/k)\}$
$\geq \{\mathcal{M}(x, y, y, t/k)*\mathcal{M}(x, y, y, t/k)\}$
$\geq \mathcal{M}(x, y, y, t/k)$
.
.
.
$\geq \mathcal{M}(x, y, y, t/k^n)$
$\to 1$ as $n \to \infty$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.
Therefore, $x = y$.
which is contradiction to $x \neq y$.
Hence $\{T_n\}$ have a unique common fixed point.
This completes the proof.

Remark 2.13: By taking $T_i = T_j = T$ in the above theorem, we get the following corollary 2.14

Corollary 2.14: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$$\mathcal{M}(Tx, Ty, Ty, t) \geq \{\mathcal{M}(x, y, y, t/k) * \mathcal{M}(x, x, Tx, t/k) * \mathcal{M}(y, y, Ty, t/k) * \mathcal{M}(x, x, Ty, 2t/k)\}$$

for all $x, y \in X$. Then T has a unique fixed point.

Theorem 2.15: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_ix, T_jy, T_jy, t) \geq \min\{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, T_ix, T_jy, 2t/k)\}$

for all $i \neq j$ and for all $x, y \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2,\ldots$
Now we prove that $\{x_n\}$ is a Cauchy sequence in X.
For $n \geq 0$, we have

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) = \mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$
$\geq \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}, 2t/k)\}$
$= \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x_n, x_{n+1}, x_{n+2}, 2t/k)\}$
$\geq \min\{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k),$
$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$
$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)*\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t/k)\}$, which implies that

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$.

Continuing this way we get

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+2}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, t/k)$
$\geq \mathcal{M}(x_{n-1}, x_n, x_n, t/k^2)$
.
.
.
$\geq \mathcal{M}(x_0, x_1, x_1, t/k^{n+1})$.

Now for any positive integer p and $t > 0$, we have

$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$$\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overbrace{*\ldots\ldots*}^{p \ \ times} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})$$

$$\geq \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^n}) \overbrace{*\ldots\ldots*}^{p \ \ times} \mathcal{M}(x_0, x_1, x_1, \frac{t}{pk^{n+p-1}})$$

Therefore, by (*FM-6*), we have

$$lim_{n\to\infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overbrace{*\ldots\ldots*}^{p \ \ times} 1 = 1$$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.
Now we prove that x is a fixed point of $\{T_n\}$ for all n.
Now we have

$\mathcal{M}(T_mx, x, x, t) = lim_{n\to\infty}$ $\mathcal{M}(T_mx, x_{n+2}, x_{n+2}, t)$
$= lim_{n\to\infty}$ $\mathcal{M}(T_mx, T_{n+2}x_{n+1}, T_{n+2}x_{n+1}, t)$
$\geq lim_{n\to\infty}\min\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x, T_mx, T_{n+2}x_{n+1}, 2t/k)\}$
$= lim_{n\to\infty}$ $\min\{\mathcal{M}(x, x_{n+1}, x_{n+1}, t/k), \mathcal{M}(x, T_mx, x_{n+2}, 2t/k)\}$
$= \min\{\mathcal{M}(x, x, x, t/k), \mathcal{M}(x, T_mx, x, 2t/k)\}$

$= \min \{1, \mathcal{M}(T_m x, x, x, 2t/k)\}$
$= \mathcal{M}(T_m x, x, x, 2t/k)$
$\geq \mathcal{M}(T_m x, x, x, t/k)$
.
.
.
$\geq \mathcal{M}(T_m x, x, x, t/k^n)$
$\to 1$ as $n \to \infty$
Hence $\mathcal{M}(T_m x, x, x, t) = 1$, for all $t > 0$.
Therefore, $T_m x = x$.
Hence $T_n x = x$ for all n.
Therefore x is a common fixed point of $\{T_n\}$.
Uniqueness: Suppose $x \neq y$ such that $T_n y = y$ for all n.
Then
$\mathcal{M}(x, y, y, t) = \mathcal{M}(T_i x, T_j y, T_j y, t)$
$\geq \min \{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, T_i x, T_j y, 2t/k)\}$
$= \min \{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, x, y, 2t/k)\}$
$= \min \{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, y, y, 2t/k)\}$
$= \mathcal{M}(x, y, y, t/k)$
.
.
.
$\geq \mathcal{M}(x, y, y, t/k^n)$
$\to 1$ as $n \to \infty$
Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.
Therefore, $x = y$.
which is contradiction to $x \neq y$.
Hence $\{T_n\}$ have a unique common fixed point.
This completes the proof.

Remark 2.16: By taking $T_i = T_j = T$ in the above theorem, we get the following corollary 2.17

Corollary 2.17: Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$ – fuzzy metric space with continuous t-norm $*$ is defined by $a*b = \min\{a, b\}$ and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition
$\mathcal{M}(Tx, Ty, Ty, t) \geq \min \{\mathcal{M}(x, y, y, t/k), \mathcal{M}(x, Tx, Ty, 2t/k)\}$
for all $x, y \in X$. Then T has a unique fixed point.

Theorem 2.18: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition
$3\mathcal{M}(T_i x, T_j y, T_k z, t) \geq \{\mathcal{M}(x, y, z, t/k) + \mathcal{M}(x, T_i x, T_j y, t/k) + ½[\mathcal{M}(y, T_j y, T_k z, t/k) + \mathcal{M}(z, T_k z, T_i x, t/k)]\}$
for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.
Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1} x_n$ for $n = 0, 1, 2, \ldots$
Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have
$3\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$
$= 3\mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$
$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}, t/k) + ½[\mathcal{M}(x_{n+1}, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t/k) + \mathcal{M}(x_{n+2}, T_{n+3}x_{n+2}, T_{n+1}x_n, t/k)]\}$
$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
$+ ½[\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + \mathcal{M}(x_{n+2}, x_{n+3}, x_{n+1}, t/k)]\}$
$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
$+ \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)\}$
$= 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$
$\geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$
Therefore,
$2\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
That is, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$.
Continuing this way we get
$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
$\geq \mathcal{M}(x_{n-1}, x_n, x_{n+1}, t/k^2)$
.
.
.
$\geq \mathcal{M}(x_0, x_1, x_2, t/k^{n+1})$
$\to 1$ as $n \to \infty$
Since $\mathcal{M}$ is first type, we have
$\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \geq \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$
Therefore, $\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \to 1$ as $n \to \infty$
Now for any positive integer p and $t > 0$, we have
$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$$\geq \mathcal{M}\left(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}\right) \overset{p\ times}{*\ldots\ldots*} \mathcal{M}\left(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p}\right)$$

Taking limit as $n \to \infty$ we get

$$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overset{p\ times}{*\ldots\ldots*} 1 = 1$$

Therefore, $lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) = 1$
which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.
Now we prove that x is a fixed point of $\{T_n\}$ for all n.
Now we have
$3\mathcal{M}(T_m x, x, x, t) = lim_{n \to \infty}\ 3\mathcal{M}(T_m x, x_{n+2}, x_{n+3}, t)$
$= lim_{n \to \infty}\ 3\mathcal{M}(T_m x, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$
$\geq lim_{n \to \infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x, T_m x, T_{n+2}x_{n+1}, t/k) + ½[\mathcal{M}(x_{n+1}, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t/k) + \mathcal{M}(x_{n+2}, T_{n+3}x_{n+2}, T_m x, t/k)]\}$
$= lim_{n \to \infty} \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x, T_m x, x_{n+2}, t/k)$
$+ ½[\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + \mathcal{M}(x_{n+2}, x_{n+3}, T_m x, t/k)]\}$
$= \{\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, T_m x, x, t/k)$
$+ ½ [\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, x, T_m x, t/k)]\}$
$= \{1 + \mathcal{M}(T_m x, x, x, t/k) + ½[1 + \mathcal{M}(T_m x, x, x, t/k)]\}$

$= \frac{1}{2}[3\mathcal{M}(T_mx, x, x, t/k) + 3]$

$6\mathcal{M}(T_mx, x, x, t) \geq 3\mathcal{M}(T_mx, x, x, t/k) + 3$

$\geq 3\mathcal{M}(T_mx, x, x, t) + 3$

Therefore, $3\mathcal{M}(T_mx, x, x, t) \geq 3$

That is, $\mathcal{M}(T_mx, x, x, t) \geq 1$

Hence $\mathcal{M}(T_mx, x, x, t) = 1$, for all $t > 0$.

Therefore, $T_mx = x$.

Hence $T_nx = x$ for all n.

Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_ny = y$ for all n. Then

$3\mathcal{M}(x, y, y, t) = 3\mathcal{M}(T_ix, T_jy, T_ky, t)$

$\geq \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, T_ix, T_jy, t/k)$

$+ \frac{1}{2}[\mathcal{M}(y, T_jy, T_ky, t/k) + \mathcal{M}(y, T_ky, T_ix, t/k)]\}$

$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, y, t/k)$

$+ \frac{1}{2}[\mathcal{M}(y, y, y, t/k) + \mathcal{M}(y, y, x, t/k)]\}$

$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, y, y, t/k)$

$+ \frac{1}{2}[1 + \mathcal{M}(x, y, y, t/k)]\}$

$= \frac{1}{2}[5\mathcal{M}(x, y, y, t/k) + 1]$

$6\mathcal{M}(x, y, y, t) \geq 5\mathcal{M}(x, y, y, t/k) + 1$

$\geq 5\mathcal{M}(x, y, y, t) + 1$

Therefore, $\mathcal{M}(x, y, y, t) \geq 1$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.

Therefore, $x = y$.

which is contradiction to $x \neq y$.

Hence $\{T_n\}$ have a unique common fixed point. This completes the proof.

Remark 2.19: From the above theorem we have,

$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \frac{1}{3}\{\mathcal{M}(x, y, z, t/k) + \mathcal{M}(x, T_ix, T_jy, t/k) + \frac{1}{2}[\mathcal{M}(y, T_jy, T_kz, t/k) + \mathcal{M}(z, T_kz, T_ix, t/k)]\}$

$\geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, T_ix, T_jy, t/k), \frac{1}{2}[\mathcal{M}(y, T_jy, T_kz, t/k) + \mathcal{M}(z, T_kz, T_ix, t/k)]\}$

Therefore,

$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, T_ix, T_jy, t/k), \frac{1}{2}[\mathcal{M}(y, T_jy, T_kz, t/k) + \mathcal{M}(z, T_kz, T_ix, t/k)]\}$

Hence we get the following corollary.

Corollary 2.20: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$– fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, T_ix, T_jy, t/k), \frac{1}{2}[\mathcal{M}(y, T_jy, T_kz, t/k) + \mathcal{M}(z, T_kz, T_ix, t/k)]\}$

for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.

Remark 2.21: By taking $T_i = T_j = T_k = T$ in the above corollary, we get the following corollary 2.22

Corollary 2.22: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$– fuzzy metric space and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, Tx, Ty, t/k), \frac{1}{2}[\mathcal{M}(y, Ty, Tz, t/k) + \mathcal{M}(z, Tz, Tx, t/k)]\}$

for all $x, y, z \in X$. Then T has a unique fixed point.

Theorem 2.23: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$– fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$3\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \{\mathcal{M}(x, y, z, t/k) + \mathcal{M}(y, z, T_kz, t/k) + \frac{1}{2}[\mathcal{M}(x, T_ix, z, t/k) + \mathcal{M}(x, y, T_jy, t/k)]\}$

for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2, \ldots$

Now we prove that $\{x_n\}$ is a Cauchy sequence in X.

For $n \geq 0$, we have

$3\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$

$= 3\mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$

$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, T_{n+3}x_{n+2}, t/k)$

$+ \frac{1}{2}[\mathcal{M}(x_n, T_{n+1}x_n, x_{n+2}, t/k) + \mathcal{M}(x_n, x_{n+1}, T_{n+2}x_{n+1}, t/k)]\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$

$+ \frac{1}{2}[\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)]\}$

$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$

$+ \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)\}$

$= 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$

$\geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$

Therefore,

$2\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq 2\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$

That is, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$.

Continuing this way we get

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$

$\geq \mathcal{M}(x_{n-1}, x_n, x_{n+1}, t/k^2)$

$\vdots$

$\geq \mathcal{M}(x_0, x_1, x_2, t/k^{n+1})$

$\to 1$ as $n \to \infty$

Since $\mathcal{M}$ is first type, we have

$\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \geq \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \to 1$ as $n \to \infty$

Now for any positive integer p and $t > 0$, we have

$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overbrace{* \ldots\ldots *}^{p\ times} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})$

Taking limit as $n \to \infty$ we get

$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overbrace{* \ldots\ldots *}^{p\ times} 1 = 1$

Therefore, $lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) = 1$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.
Now we prove that x is a fixed point of $\{T_n\}$ for all n.
Now we have

$3\mathcal{M}(T_mx, x, x, t) = lim_{n \to \infty}\ 3\mathcal{M}(T_mx, x_{n+2}, x_{n+3}, t)$
$= lim_{n \to \infty}\ 3\mathcal{M}(T_mx, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$
$\geq lim_{n \to \infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, T_{n+3}x_{n+2}, t/k) + ½[\mathcal{M}(x, T_mx, x_{n+2}, t/k)+\mathcal{M}(x, x_{n+1}, T_{n+2}x_{n+1}, t/k)]\}$
$= lim_{n \to \infty}\ \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + ½[\mathcal{M}(x, T_mx, x_{n+2}, t/k)+\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k)]\}$
$= \{\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, x, x, t/k) + ½[\mathcal{M}(x, T_mx, x, t/k)+\mathcal{M}(x, x, x, t/k)]\}$
$= \{1 + 1 + ½[\mathcal{M}(T_mx, x, x, t/k)+1]\}$
$= ½[\mathcal{M}(T_mx, x, x, t/k) + 5]$
$6\mathcal{M}(T_mx, x, x, t) \geq \mathcal{M}(T_mx, x, x, t/k) + 5$
$\geq \mathcal{M}(T_mx, x, x, t) + 5$

Therefore, $5\mathcal{M}(T_mx, x, x, t) \geq 5$
That is, $\mathcal{M}(T_mx, x, x, t) \geq 1$
Hence $\mathcal{M}(T_mx, x, x, t) = 1$, for all $t > 0$.
Therefore, $T_mx = x$.
Hence $T_nx = x$ for all n.
Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_ny = y$ for all n. Then

$3\mathcal{M}(x, y, y, t) = 3\mathcal{M}(T_ix, T_jy, T_ky, t)$
$\geq \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(y, y, T_ky, t/k) + ½[\mathcal{M}(x, T_ix, y, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$
$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(y, y, y, t/k) + ½[\mathcal{M}(x, x, y, t/k)+\mathcal{M}(x, y, y, t/k)]\}$
$= \{\mathcal{M}(x, y, y, t/k) + 1 + ½[\mathcal{M}(x, y, y, t/k)+\mathcal{M}(x, y, y, t/k)]\}$
$3\mathcal{M}(x, y, y, t) \geq 2\mathcal{M}(x, y, y, t/k) + 1$
$\geq 2\mathcal{M}(x, y, y, t) + 1$

Therefore, $\mathcal{M}(x, y, y, t) \geq 1$
Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.
Therefore, $x = y$. which is contradiction to $x \neq y$.
Hence $\{T_n\}$ have a unique common fixed point.
This completes the proof.

Remark 2.24: From the above theorem we have,

$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \frac{1}{3}\{\mathcal{M}(x, y, z, t/k) + \mathcal{M}(y, z, T_kz, t/k) + ½[\mathcal{M}(x, T_ix, z, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$
$\geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(y, z, T_kz, t/k), ½[\mathcal{M}(x, T_ix, z, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$

Therefore,
$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(y, z, T_kz, t/k), ½[\mathcal{M}(x, T_ix, z, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$
Hence we get the following corollary.

Corollary 2.25: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and T_n: $X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition
$\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(y, z, T_kz, t/k), ½[\mathcal{M}(x, T_ix, z, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$
for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.

Remark 2.26: By taking $T_i = T_j = T_k = T$ in the above corollary, we get the following corollary 2.27

Corollary 2.27: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and let T: $X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition
$\mathcal{M}(Tx, Ty, Tz, t) \geq \min\{\mathcal{M}(x, y, z, t/k), \mathcal{M}(y, z, Tz, t/k), ½[\mathcal{M}(x, Tx, z, t/k)+\mathcal{M}(x, y, Ty, t/k)]\}$
for all $x, y, z \in X$. Then T has a unique fixed point.

Theorem 2.28: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and T_n: $X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition
$5\mathcal{M}(T_ix, T_jy, T_kz, t) \geq \{\mathcal{M}(x, y, z, t/k) + \mathcal{M}(x, T_ix, T_jy, t/k) + \mathcal{M}(y, z, T_kz, t/k) + ½[\mathcal{M}(y, T_jy, T_kz, t/k)+\mathcal{M}(z, T_kz, T_ix, t/k)] + ½[\mathcal{M}(x, T_ix, z, t/k)+\mathcal{M}(x, y, T_jy, t/k)]\}$
for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be any arbitrary element.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{n+1}x_n$ for $n = 0, 1, 2,\ldots$
Now we prove that $\{x_n\}$ is a Cauchy sequence in X.
For $n \geq 0$, we have

$5\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$
$= 5\mathcal{M}(T_{n+1}x_n, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$
$\geq \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, T_{n+1}x_n, T_{n+2}x_{n+1}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, T_{n+3}x_{n+2}, t/k) + ½[\mathcal{M}(x_{n+1}, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t/k)+\mathcal{M}(x_{n+2}, T_{n+3}x_{n+2}, T_{n+1}x_n, t/k) + ½[\mathcal{M}(x_n, T_{n+1}x_n, x_{n+2}, t/k)+\mathcal{M}(x_n, x_{n+1}, T_{n+2}x_{n+1}, t/k)]\}$
$=\{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)+\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) +\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + ½[\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)+\mathcal{M}(x_{n+2}, x_{n+3}, x_{n+1}, t/k)] + ½[\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)+\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)]\}$
$= \{\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)\}$
$= 3\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + 2\ \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$
$\geq 3\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k) + 2\ \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$

Therefore,
$3\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq 3\mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
That is, $\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$.

Continuing this way we get

$\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t) \geq \mathcal{M}(x_n, x_{n+1}, x_{n+2}, t/k)$
$\geq \mathcal{M}(x_{n-1}, x_n, x_{n+1}, t/k^2)$
.
.
.
$\geq \mathcal{M}(x_0, x_1, x_2, t/k^{n+1})$
$\to 1$ as $n \to \infty$

Since $\mathcal{M}$ is first type, we have

$\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \geq \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t)$

Therefore, $\mathcal{M}(x_{n+1}, x_{n+1}, x_{n+2}, t) \to 1$ as $n \to \infty$

Now for any positive integer p and $t > 0$, we have

$\mathcal{M}(x_n, x_{n+p}, x_{n+p}, t)$

$\geq \mathcal{M}(x_n, x_{n+1}, x_{n+1}, \frac{t}{p}) \overbrace{* \ldots\ldots *}^{p \ times} \mathcal{M}(x_{n+p-1}, x_{n+p}, x_{n+p}, \frac{t}{p})$

Taking limit as $n \to \infty$ we get

$lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) \geq 1 \overbrace{* \ldots\ldots *}^{p \ times} 1 = 1$

Therefore, $lim_{n \to \infty} \mathcal{M}(x_n, x_{n+p}, x_{n+p}, t) = 1$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X. Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Now we prove that x is a fixed point of $\{T_n\}$ for all n.

Now we have

$5\mathcal{M}(T_m x, x, x, t) = lim_{n \to \infty}\ 5\mathcal{M}(T_m x, x_{n+2}, x_{n+3}, t)$

$= lim_{n \to \infty}\ 5\mathcal{M}(T_m x, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t)$

$\geq lim_{n \to \infty} \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k) + \mathcal{M}(x, T_m x, T_{n+2}x_{n+1}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, T_{n+3}x_{n+2}, t/k) + ½[\mathcal{M}(x_{n+1}, T_{n+2}x_{n+1}, T_{n+3}x_{n+2}, t/k) + \mathcal{M}(x_{n+2}, T_{n+3}x_{n+2}, T_m x, t/k)] + ½[\mathcal{M}(x, T_m x, x_{n+2}, t/k) + \mathcal{M}(x, x_{n+1}, T_{n+2}x_{n+1}, t/k)]\}$

$= lim_{n \to \infty} \{\mathcal{M}(x, x_{n+1}, x_{n+2}, t/k)$
$+ \mathcal{M}(x, T_m x, x_{n+2}, t/k) + \mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k)$
$+ ½[\mathcal{M}(x_{n+1}, x_{n+2}, x_{n+3}, t/k) + \mathcal{M}(x_{n+2}, x_{n+3}, T_m x, t/k)]$
$+ ½[\mathcal{M}(x, T_m x, x_{n+2}, t/k) + \mathcal{M}(x, x_{n+1}, x_{n+2}, t/k)]\}$

$= \{\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, T_m x, x, t/k) + \mathcal{M}(x, x, x, t/k)$
$+ ½[\mathcal{M}(x, x, x, t/k) + \mathcal{M}(x, x, T_m x, t/k)]$
$+ ½[\mathcal{M}(x, T_m x, x, t/k) + \mathcal{M}(x, x, x, t/k)]\}$

$= \{1 + \mathcal{M}(T_m x, x, x, t/k) + 1 + ½[1 + \mathcal{M}(T_m x, x, x, t/k)]$
$+ ½[\mathcal{M}(T_m x, x, x, t/k) + 1]\}$

$= 3 + 2\mathcal{M}(T_m x, x, x, t/k)$

$\geq 3 + 2\mathcal{M}(T_m x, x, x, t)$

Therefore, $3\mathcal{M}(T_m x, x, x, t) \geq 3$

That is, $\mathcal{M}(T_m x, x, x, t) \geq 1$

Hence $\mathcal{M}(T_m x, x, x, t) = 1$, for all $t > 0$.

Therefore, $T_m x = x$.

Hence $T_n x = x$ for all n.

Therefore x is a common fixed point of $\{T_n\}$.

Uniqueness: Suppose $x \neq y$ such that $T_n y = y$ for all n. Then

$5\mathcal{M}(x, y, y, t) = 5\mathcal{M}(T_i x, T_j y, T_k y, t)$

$\geq \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, T_i x, T_j y, t/k)$
$+ \mathcal{M}(y, y, T_k y, t/k)$
$+ ½[\mathcal{M}(y, T_j y, T_k y, t/k) + \mathcal{M}(y, T_k y, T_i x, t/k)]$
$+ ½[\mathcal{M}(x, T_i x, y, t/k) + \mathcal{M}(x, y, T_j y, t/k)]\}$

$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, x, y, t/k) + \mathcal{M}(y, y, y, t/k)$
$+ ½[\mathcal{M}(y, y, y, t/k) + \mathcal{M}(y, y, x, t/k)]$
$+ ½[\mathcal{M}(x, x, y, t/k) + \mathcal{M}(x, y, y, t/k)]\}$

$= \{\mathcal{M}(x, y, y, t/k) + \mathcal{M}(x, y, y, t/k) + 1$
$+ ½[1 + \mathcal{M}(x, y, y, t/k)] + \mathcal{M}(x, y, y, t/k)\}$

$= ½[7\mathcal{M}(x, y, y, t/k) + 3]$

$10\mathcal{M}(x, y, y, t) \geq 7\mathcal{M}(x, y, y, t/k) + 3$
$\geq 7\mathcal{M}(x, y, y, t) + 3$

Therefore, $3\mathcal{M}(x, y, y, t) \geq 3$

That is, $\mathcal{M}(x, y, y, t) \geq 1$

Hence $\mathcal{M}(x, y, y, t) = 1$, for all $t > 0$.

Therefore, $x = y$.

which is contradiction to $x \neq y$.

Hence $\{T_n\}$ have a unique common fixed point.

This completes the proof.

Corollary 2.29: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and $T_n: X \to X$ be a sequence of maps such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(T_i x, T_j y, T_k z, t) \geq \min \{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, T_i x, T_j y, t/k), \mathcal{M}(y, z, T_k z, t/k), ½[\mathcal{M}(y, T_j y, T_k z, t/k) + \mathcal{M}(z, T_k z, T_i x, t/k)], ½[\mathcal{M}(x, T_i x, z, t/k) + \mathcal{M}(x, y, T_j y, t/k)]\}$

for all $i \neq j \neq k$ and for all $x, y, z \in X$. Then $\{T_n\}$ have a unique common fixed point.

Remark 2.30: By taking $T_i = T_j = T_k = T$ in the above corollary, we get the following corollary 2.31

Corollary 2.31: Let $(X, \mathcal{M}, *)$ be a complete first type $\mathcal{M}$ – fuzzy metric space and let $T: X \to X$ be a mapping such that for all $t > 0$ and $0 < k < 1$ satisfying the condition

$\mathcal{M}(Tx, Ty, Tz, t) \geq \min \{\mathcal{M}(x, y, z, t/k), \mathcal{M}(x, Tx, Ty, t/k), \mathcal{M}(y, z, Tz, t/k), ½[\mathcal{M}(y, Ty, Tz, t/k) + \mathcal{M}(z, Tz, Tx, t/k)], ½[\mathcal{M}(x, Tx, z, t/k) + \mathcal{M}(x, y, Ty, t/k)]\}$

for all $x, y, z \in X$. Then T has a unique fixed point.

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