On The Extension of Topological Local Groups with Local Cross Section

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Research Article

Abstract: In this paper, we introduce the cohomology of topological local groups and topological local extensions. We show that the second cohomology of a local topological group is in one to one correspondence with the class of topological local extensions with local cross sections.

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1.Introduction:

Let H and G be topological groups, H abelian. We consider H as a G-module with a continuous action of G on H, that is, a continuous function from $G\times H$ into H, carrying (g,h) onto gh.

By a topological extension of H by G, $\varepsilon=(E,\pi)$, we mean a short exact sequence

$\varepsilon:1\longrightarrow H\xrightarrow{\iota}E\xrightarrow{\pi}G\longrightarrow 1$ with π an open continuous homomorphism and H a closed normal subgroup of E.

Two extensions, $\varepsilon_1=(E_1,\pi_1)$, and $\varepsilon_2=(E_2,\pi_2)$, of H by G are said to be equivalent, $\varepsilon_1\equiv\varepsilon_2$, if there exists a continuous isomorphism $\sigma:E_1\to E_2$ such that $\sigma\iota_1=\iota_2$ and $\pi_1=\pi_2\sigma$. The set of equivalence classes of extensions of H by G, denoted by $Ext_c(G,H)$, with the Baire-sum is a group [2].

A cross-section of a topological group extension (E,π) of H by G is a continuous map $u:G\to E$ such that $\pi u(x)=x$ for each $x\in G$. There is a one to one correspondence between $Ext_c(G,H)$, and $H^2(G,H)$ [2].

In this paper we show a similar result for topological local groups [3]. In section 1 we give some primarily definitions which will be needed in sequel. In section 2, we introduce the local extension on topological local groups and prove that the second cohomology of topological local group is isomorphic with the group of equivalence classes of topological local extensions with local crossed-sections.

We use the following notations:

- " 1 " is the identity element of X.
- "$\leq$" : $G\leq H$, G a sublocal group (subgroup) of a local group (group) H.
- $D=\{(x,y)\in X\times X; xy\in X\}$ where X is a local group.

2.Primary Definitions:

We recall the following definition from [5]:

A *local group* $(X,.)$ is like a group except that the action of group is not necessarily defined for all pairs of elements, The associative law takes the following form: if $x.y$ and $y.z$ are defined, then if one of the products $(x.y).z$, $x.(y.z)$ is defined, so is the other and the two products are equal. It is assumed that each element of X has an inverse.

Definition 2.1 [3] *Let X be a local group, if there exist:*

a) a distinguished element $e\in X$, the identity element,

b) a continuous product map $\varphi:D\to X$ defined on an open subset
$$(e\times X)\cup(X\times e)\subset D\subset X\times X.$$

c) a continuous inversion map $\nu:X\to X$

satisfying the following properties:

(i)*Identity*: $\varphi(e,x)=x=\varphi(x,e)$ for every $x\in X$

(ii) *Inverse*: $\varphi(\nu(x),x)=e=\varphi(x,\nu(x))$ for every $x\in X$

(iii) *Associativity*: If (x,y), (y,z), $(\varphi(x,y),z)$ and $(x,\varphi(y,z))$ all belong to D, then
$$\varphi(\varphi(x,y),z)=\varphi(x,\varphi(y,z))$$

then X is called a *topological local group*.

Example 2.2*Let X be a Hausdorff topological space and Δ_X be the diagonal of X, $a\in X$ and $D=(\{a\}\times X)\cup(X\times\{a\})\cup\Delta_X$. Define*

$\varphi : D \to X$ *by:*

$$\varphi(x,y) = \begin{cases} x & ,y = a, \\ y & ,x = a, \\ a & ,x = y, \end{cases} .$$

Now X, by the action of φ, is a local group.

If $x \in X$, $x \neq a$, we have $\varphi(x,a) = x$. If U is a neighborhood of x, then $\varphi^{-1}(U) = U \times \{a\}$. There are two cases;

1) $a \in U$: since X is Hausdorff, there are disjoint neighborhood U_1, U_2 containing a, x, respectively. Then $x \in U_2 \cap U$ and $a \notin U_2 \cap U = V$ and $\varphi^{-1}(V) = V \times \{a\}$. Hence, $\varphi(V \times \{a\}) \subset U$. So φ is continuous.
2) $a \notin U$: $\varphi^{-1}(U) = U \times \{a\}$.

If $x = a$ and W is a closed neighborhood of a in X then $\varphi^{-1}(W) = \Delta_X \cup (W \times \{a\}) \cup (\{a\} \times W)$. Hence, φ is continuous. Therefore, $\varphi : D \to X$, $(x,y) \mapsto xy$ and $X \to X$, $x \mapsto x^{-1}$ are continuous. So X is a topological local group.

Definition 2.3 *Let X and Y be topological local groups. We say that X operates on the left of Y if:* There is a neighborhood X_1 of the identity in X and a neighborhood Y_1 of the identity in Y such that for every $x \in X_1$, $y \in Y_1$ there exists $xy \in Y$ with the following condition:

1. $X_1 \times Y_1 \to Y$, $(x,y) \mapsto xy$ is continuous;
2. $1.y = y$ for all $y \in Y_1$;
3. If $y_1, y_2 \in Y_1$ and $y_1 y_2 \in Y_1$ and $xy_1 \in Y_1$ is defined for all $x \in X_1$ then $x(y_1 y_2) = (xy_1)y_2$;
4. If $x_1, x_2 \in X_1$, $y \in Y_1$ are so that $x_2 y$ and $x_1 x_2$ are defined in Y_1 and X_1 respectively, then $x_1(x_2 y) = (x_1 x_2)y$.

Definition 2.4 *A continuous map $f : (X,.) \to (X',*)$ of topological local groups, is called a homomorphism if;*

1. $(f \times f)(D) \subseteq D'$ where $D' = \{(x',y') \in X' \times X', x' * y' \in X'\}$;
2. $f(e) = e'$ and $f(x^{-1}) = (f(x))^{-1}$;
3. if $x.y \in X$ then $f(x) * f(y)$ exists in X' and $f(x.y) = f(x) * f(y)$.

With these morphisms topological local groups form a category which contains the subcategory of topological groups.

Definition 2.5 *A homomorphism of topological local groups $f : (X,.) \to (X',*)$ is called strong if for every $x, y \in X$, the existence of $f(x) * f(y)$ implies that $x.y \in X$.*

A morphism is called a monomorphism (epimorphism) if it is injective (surjective).

Lemma 2.6 *[1, Lemma 2.5] Let U be a symmetric neighborhood of the identity in a topological local group X. There is a neighborhood U_0 of identity in U such that for every $x, y \in U$, $xy \in U_0$.*

We denote the product of p copies of X by X^p.

Definition 2.7 *Let X and Y be topological local groups. A local p-map of X into Y is a continuous map $f : V^p \to Y$ where V is a symmetric neighborhood of identity in X such that $f(x_1,...,x_p) = 0$ whenever $x_1 = ... = x_p = 1$.*

Definition 2.8 *Let X and Y be topological local groups. Two local p-maps $f_1 : V_1^p \to Y$ and $f_2 : V_2^p \to Y$ of X into Y, where V_1, V_2 are symmetric neighborhoods of the identity in X, are said to be equivalent if there is a neighborhood V with $V \subseteq V_1 \cap V_2$ such that*

$$f_1(x_1,...,x_p) = f_2(x_1,...,x_p)$$

whenever $x_i \in V$ for all $i \in \{1,...,p\}$.

Definition 2.9 *The equivalence class of a local p-map is called a local p-cochain of X to Y.*

Let X and Y be topological local groups and Y abelian (written additively). Let $C_L^p(X,Y)$ be the set of equivalence classes of local p-maps, with the usual addition of functions. The set $C_L^p(X,Y)$ is an abelian group. Therefor, we define an addition on $C_L^p(X,Y)$. Suppose $[f_1],[f_2] \in C_L^p(X,Y)$ and V_1, V_2 are symmetric neighborhoods of the identity in X where $f_1 : V_1^p \to Y$, $f_2 : V_2^p \to Y$ are local p-maps. Let U_1 be a neighborhood of identity in Y. By Lemma 2.6, there is a symmetric neighborhoods U_0 in U_1 such that $y_1 + y_2 \in U_0$ is defined when $y_1, y_2 \in U_1$. Since f_1, f_2 are continuous, then there exists a neighborhood V of 1 in X such that $V^p \subset f_1^{-1}(U_0) \cap f_2^{-1}(U_0)$. Now define a local map $f : V^p \to Y$ by

$$f(x_1,...,x_p) = f_1(x_1,...,x_p) + f_2(x_1,...,x_p)$$

for every $x_i \in V$ and $i \in \{1,...,p\}$. It is clear that the local p-cochain $[f]$ does not depend on the choice of the representations f_1 and f_2. Hence, we define an addition in $C_L^p(X,Y)$, by

$$[f(x_1,...,x_p)] = [f_1(x_1,...,x_p)] + [f_2(x_1,...,x_p)].$$

Definition 2.10 *Suppose X and Y are topological local groups and Y abelian. We define a coboundary operator*

$$\delta : C_L^p(X,Y) \to C_L^{p+1}(X,Y).$$

Let U_1 be a neighborhood of the identity in Y. By Lemma 2.6, there is a neighborhood U_0 of the identity in Y, $U_0 \subseteq U_1$ such that $\Sigma_{i=0}^{p+1}(-1)^i y_i$ is defined whenever $y_i \in U_1$. Suppose V_0 is a neighborhood of 1 in X such that $xy \in U_1$ whenever $x \in V_0$ and $y \in U_0$. Suppose $[f] \in C_L^p(X,Y)$, $f : V_0^p \to Y$ a local p-map. By continuity of f, we choose a symmetric neighborhood V_1 in X such that $V_1^p \subset f^{-1}(U_0)$. By Lemma 2.6, there is a symmetric neighborhood V_2 in X such that $V_2 \subseteq V_1 \cap V_0$, $x_i x_j \in V_2$ for all $x_i, x_j \in V_1 \cap V_0$.

Define a local (p+1)-map $\delta f : V_2^{p+1} \to Y$, for each point $(x_1,...,x_{p+1}) \in V_2^{p+1}$ by

$$\delta f(x_1,...,x_{p+1}) = x_1 f(x_2,...,x_{p+1})$$
$$+ \Sigma_{i=0}^{p}(-1)^p f(x_1,...,x_i x_{i+1}, x_{p+1}) + (-1)^{p+1} f(x_1,...,x_p)$$

It is easy to show that the local (p+1)-cochain $[\delta f]$ depends only on the given local p-cochain $[f]$ and $\delta[f] = [\delta f]$.

Definition 2.11 *A local p-cochains $[f]$ such that $\delta[f] = 0$ is called a local p-cocycles.*

We denote the set of all p-cocyles by $Z_L^p(X,Y)$.

Definition 2.12 *The image of a coboundry operator in $C_L^{p+1}(X,Y)$ is called a local p-coboundry. We denote the set of all p-coboundries by $B_L^p(X,Y)$.*

It is easy to show that $B_L^p(X,Y) \subseteq Z_L^p(X,Y) \subseteq C_L^p(X,Y)$,since $\delta\delta[f] = 0$. Then $B_L^p(X,Y)$ is a subgroup of $Z_L^p(X,Y)$.

Definition 2.13 *Let X and Y be topological local groups and Y abelian. Then $H_L^p(X,Y) = \dfrac{Z_L^p(X,Y)}{B_L^p(X,Y)}$ is called the p-th cohomology topological local group.*

3.Second Cohomology and Topological Local Group Extensions:

In this part we prove that the second cohomology of a topological local group is isomorphic with the group of the equivalence classes of topological local extensions with local crossed-sections.

Definition 3.1 *Let X, Y be topological local groups and U a symmetric neighborhood in X. The continuous map $f : U \to Y$ is an open continuous local homomorphism of X onto Y if*

1. there exists a symmetric neighborhood U_0 in U which $x_1, x_2 \in U$, $x_1 x_2 \in U_0$;
2. $f(x_1 x_2) = f(x_1) f(x_2)$, $x_1 x_2 \in U_0$;
3. for every symmetric neighborhood W, $W \subseteq U_0$, $f(W)$ is open in Y.

The map f is called an *open continuous local isomorphism* of X to Y if U_0 can be chosen that so $f|_{U_0}$ is one to one.

Definition 3.2 *A topological local group extension of the topological local group Y by a topological local group X is a triple (E, π, η) where E is a topological local group, π is an open continuous local homomorphism of E to X, and η is an open continuous local isomorphism of Y onto the kernel of π [2].*

Remark 3.3*If (E, π, η) is a topological local group extension of N by X, where π a strong homomorphism and $N = ker\pi$, then N is a closed normal topological subgroup of E.*

Definition 3.4 *Let (E, π, η) be a topological local group extension of Y by X. A continuous map $u : V \to E$ where V is a neighborhood of 1 in X is called a local cross-section if $\pi u(x) = x$ for every $x \in V$.*

Definition 3.5*A topological local group extension (E, π, η) of X is said to be fibered if it has a local cross section.*

Definition 3.6*A topological local group extension (E, π, η) of X is said to be essential if it has a* *local homomorphism of* X *to* E.

Definition 3.7 *Let* $\varepsilon_1 = (E_1, \pi_1, \eta_1)$ *and* $\varepsilon_2 = (E_2, \pi_2, \eta_2)$ *be topological local extensions of an abelian topological local group* C *by a topological local group* X. *If there exists a strong isomorphism* σ *of* E_1 *onto* E_2 *such that* $\sigma \circ \eta_1(n) = \eta_2(n)$ *and* $\pi_1 = \pi_2 \circ \sigma$.

$$\begin{array}{lccccccccc} \varepsilon_1: & 1 & \longrightarrow & C & \longrightarrow & E_1 & \xrightarrow{\pi_1} & X & \longrightarrow & 1 \\ & & & \downarrow & & \downarrow\sigma & & \downarrow & & \\ \varepsilon_2: & 1 & \longrightarrow & C & \longrightarrow & E_2 & \xrightarrow{\pi_2} & X & \longrightarrow & 1 \end{array}$$

then ε_1 and ε_2 are *equivalent*, $\varepsilon_1 \equiv \varepsilon_2$.

Note 3.8 *Let* E *be a topological local group. The set* C *is called the center of* E *if*

$$C = \{x \in E : \exists U \text{ symmetric neighborhood in } E, \quad xy = yx, \quad \forall y \in U\}.$$

In this paper, we replace Y by C in Definition 3.2. In this case, C is a subset of Y and $ker\pi \subseteq C$.

Definition 3.9 *Let* C *be an abelian topological local group. A pair factor set on topological local group* X *is a pair* (f, θ) *where* $f: V_1 \times V_1 \to C$ *is continuous, and* V_1 *is a neighborhood of the identity in* X. *Suppose there exist a neighborhoods* V_0 *of the identity in* X *and* $V_0 \subseteq V_1$ *such that* $x_1x_2 \in V_0$, *for every* $x_1, x_2 \in V_1$. *Let* W_0 *be a symmetric neighborhood in* C *such that* $xc \in C$, *for every* $x \in V_0$ *and* $c \in W_0$.

Now, there is an action of X on C, $\theta: X \times C \to C$ such that $\theta_x: W_0 \to C$, for every $x \in V_0$ is a local inner automorphism,

$$f(x_1, x_2) f(x_1x_2, x_3) = (\theta_{x_1} f(x_2, x_3)) f(x_1, x_2x_3) \quad (3.1)$$

whenever $x_1, x_2, x_3 \in V_1$ and

$$\theta_{x_1}\theta_{x_2} = f(x_1, x_2)\theta_{x_1x_2} f(x_1, x_2)^{-1} \quad (3.2)$$

for $x_1, x_2 \in V_1$.

Definition 3.10 *The pair factor set* (f, θ) *is normalized if*

$$\theta(1) = Id_C, \qquad f(1, x) = f(x, 1) = 1 \ \forall x \in X$$

where Id_C is the identity automorphism.

Remark 3.11 *Let* (E, π, η) *be an extension of* Y *by* X *and* (f, θ) *a pair factor set on* X *to* C *where* C *is locally isomorphism with the kernel* π. *It is clear that* $(f \circ \pi \times \pi, \theta \circ \pi)$ *is a pair factor set on* E *to* C. *We call* $(f \circ \pi \times \pi, \theta \circ \pi)$ *the extension of* (f, θ) *by* π.

Let (E, π, η) be an extension of Y by X. The local cross-section u of (E, π, η) is called *normalized* if $u(1) = 1$.

In this section we assume all pair factor sets and local cross-sections are normalized.

Definition 3.12 *Let* X, Y *be topological local groups. Suppose a topological local group extension* (E, π, η) *of* C *by* X *where* C *is locally isomorphism with the kernel* π *with a local cross-section* $u: V \to E$ *where* V *is a neighborhood of the identity in* X. *There are a symmetric neighborhood* V_0, $V_0 \subseteq V$ *and a neighborhood* W_0 *of the identity in* C *such that* $\eta(xc)$ *and* $u(x)\eta(c)u(x)^{-1}$ *are defined for every* $x \in V_0$ *and* $c \in W_0$. *Then, the action of* X *on* C *is defined by*

$$\eta(xc) = u(x)\eta(c)u(x)^{-1} \quad (3.3)$$

In Definition 3.12, since C is the center of E, we can easily see that for each $\eta(c) \in C$ and $x \in V_0 \subseteq X$, the element $u(x)\eta(c)u(x)^{-1} \in C$ dose not depend on the choice of the local cross-section u.

Proposition 3.13 *Let* X *and* C *be topological local groups and* C *abelian. Suppose* (E, π, η) *is a topological local extension of* C *by* X. *Each local cross-section* u *of* (E, π, η) *determines a pair factor set* (f, θ) *on* X *to* C.

Proof. Let V_1 be a neighborhood of the identity in X. By Lemma 2.6, there is a symmetric neighborhood V_0 in V_1 such that $x_1x_2 \in V_0$ for all $x_1, x_2 \in V_1$. Consider the continuous map $f: V_1 \times V_1 \to C$, $f(x_1, x_2) = \eta^{-1}(u(x_1)u(x_2)u(x_1x_2)^{-1})$. Let $x_1x_2x_3 \in V_0$ and $(x_1x_2)x_3 = x_1(x_2x_3)$. Hence, $u(x_1)u(x_2x_3)u(x_1x_2x_3)^{-1}$ and $u(x_1x_2)u(x_3)u(x_1x_2x_3)^{-1}$ are defined in $\pi^{-1}(V_0)$. We define

$$f(x_1x_2, x_3) = \eta^{-1}(u(x_1x_2)u(x_3)u(x_1x_2x_3)^{-1});$$
$$f(x_1, x_2x_3) = \eta^{-1}(u(x_1)u(x_2x_3)u(x_1x_2x_3)^{-1}).$$

Let $\theta_x: W_0 \to C$, $c \mapsto u(x)\eta(c)u(x)$, where W_0

is a symmetric neighborhood and $\pi^{-1}(V_0) \subseteq W_0$. Then, θ_x is a local automorphism. We show that $(\eta \circ f, \theta)$ is a pair factor set. We verify the first condition of Definition 3.9. Let $x_1x_2x_3 \in V_0$ and $\eta(f(x_1,x_2))u(x_1x_2)$ and $u(x_1)u(x_2)u(x_1x_2)^{-1}$ are defined in W_0, since (x_1x_2) and $x_1x_2(x_1x_2)^{-1}$ are in V_0. Hence, the relation $f(x_1,x_2) = \eta^{-1}(u(x_1)u(x_2)u(x_1x_2)^{-1})$ implies that

$u(x_1)u(x_2) = \eta(f(x_1,x_2))u(x_1x_2)$;

$u(x_2)u(x_3) = \eta(f(x_2,x_3))u(x_2x_3)$;

$u(x_1x_2)u(x_3) = \eta(f(x_1x_2,x_3))u(x_1x_2x_3)$;

$u(x_1)u(x_2x_3) = \eta(f(x_1,x_2x_3))u(x_1x_2x_3)$.

We have

$u(x_1)(u(x_2)u(x_3)) = u(x_1)\eta(f(x_2,x_3))u(x_2x_3)$

$= u(x_1)\eta(f(x_2,x_3))u(x_1)^{-1}u(x_1)u(x_2x_3)$

$= u(x_1)\eta(f(x_2,x_3))u(x_1)^{-1}\eta(f(x_1,x_2x_3))u(x_1x_2x_3)$ (1)

The right hand side of (1) is in E, since $x_1(x_2x_3)$, $x_1x_1^{-1}x_1(x_2x_3)$, and $x_1x_1^{-1}(x_1x_2x_3)$ are defined in V_0. Similarly, we obtain

$(u(x_1)u(x_2))u(x_3) = \eta(f(x_1,x_2))u(x_1x_2)u(x_3)$

$= \eta(f(x_1,x_2))\eta(f(x_1x_2,x_3))u(x_1x_2x_3)$ (2)

By comparing (1) and (2),

$\eta(f(x_1,x_2))\eta(f(x_1x_2,x_3)) =$

$u(x_1)\eta(f(x_2,x_3))u(x_1)^{-1}\eta(f(x_1,x_2x_3))$

for $x_1,x_2,x_3 \in V_1$ which proves (3.1). Now, for $x_1,x_2 \in V_1$ and $c \in W_0$:

$u(x_1)u(x_2)\eta(c)u(x_2)^{-1}u(x_1)^{-1} =$

$\eta(f(x_1,x_2))u(x_1x_2)\eta(c)u(x_1x_2)^{-1}\eta(f(x_1,x_2))^{-1}$.

So, (3.2) holds.□

Proposition 3.14 *Let X and C be topological local groups and C abelian. Suppose (E,π,η) is a topological local extension of C by X which has a continuous local cross-section u. Then, (E,π,η) determines uniquely an element of $H_L^2(X,C)$.*

Proof. Suppose (E,π,η) is a topological local extension with a local cross-section $u : U \to E$ where U is a symmetric neighborhood of identity. Let $u' : U' \to E$ be another local cross-section of (E,π,η), U' a symmetric neighborhood in X. Suppose the pair factor set (f',θ') corresponds to u'.

We define a local cochain $h \in C_L^1(X,C)$ by $h(x) = \eta^{-1}(u'(x)u(x)^{-1})$, for $x \in U \cap U'$.

By Lemma 2.6, there is a symmetric neighborhood U_0, $U_0 \subseteq U \cap U'$ such that $x_1x_2 \in U_0$ for $x_1,x_2 \in U \cap U'$. The map π is continuous and by Lemma 2.6, there is a neighborhood V_0, $V_0 \subseteq \pi^{-1}(U_0)$ such that $u(x_1)u(x_2), u'(x_1)u'(x_2) \in V_0$ for $x_1,x_2 \in U \cap U'$.

Let $\eta(h(x))u(x)$ and $u'(x)u(x)^{-1}$ be defined in V_0, $u'(x) = \eta(h(x))u(x)$ for all $x \in U \cap U'$.

Consider the continuous map $f : U_0 \times U_0 \to C$,

$f(x_1,x_2) = \eta^{-1}(u(x_1)u(x_2)u(x_1x_2)^{-1})$.

$u'(x_1)u'(x_2) = \eta(h(x_1))u(x_1)\eta(h(x_2))u(x_2)$;

$= \eta(h(x_1))[u(x_1)\eta(h(x_2))u(x_1)^{-1}]u(x_1)u(x_2)$;

$= \eta(h(x_1))\eta(x_1h(x_2))\eta(f(x_1,x_2))u(x_1x_2)$;

by the action (3.3)

$= \eta(h(x_1))\eta(x_1h(x_2))\eta(f(x_1,x_2))\eta(h(x_1x_2))^{-1}u'(x_1x_2)$

where $x_1,x_2 \in U_0$. Since C is abelian, then

$\eta'(f'(x_1,x_2)) =$

$\eta(h(x_1)) + \eta(x_1h(x_2)) + \eta(f(x_1,x_2)) - \eta(h(x_1x_2)) + u'(x_1x_2)$

$\eta'(f'(x_1,x_2)) - \eta(f(x_1,x_2)) =$

$\eta(h(x_1)) + \eta(x_1h(x_2)) - \eta(h(x_1x_2)) + u'(x_1x_2) =$.

$\delta(\eta(h(x_1x_2)))$

$\eta'(f') - \eta(f) = \delta\eta(h)$

Therefore, the pair factor set is independent of the choice of the continuous local cross-section.□

Definition 3.15 *Let X and Y be topological local groups with the action X on Y. Suppose X_1 and Y_1 are neighborhoods of the identities in X and Y respectively such that all products are defined. Let $\theta : X \to Aut(Y)$, $\theta(x)(y) = \theta_x(y) = xyx^{-1}$ be a continuous strong homomorphism where $x \in X_1$ and $y \in Y_1$. We define $\mu : (Y \times X) \times (Y \times X) \to Y \times X$ by*

$\mu((y,x),(y',x')) = (y'\theta_{x'}(y), xx')$

for every $x,x' \in X_1$ and $y,y' \in Y_1$. The space $(Y \times X, \mu)$ is called the *semi-direct product* of topological local groups X and Y with respect to θ, denoted by $X \underset{\theta}{\times} Y$.

Proposition 3.16 *Let X and C be topological* *local groups and C abelian. Let (f,θ) be a pair factor set on X to C. There exists an extension (E,π,η) of C by X with a continuous local cross-section u which corresponds to (f,θ).*

Proof. Let X and C be topological local groups and C abelian. Let V_1 is a neighborhood of the identity in X. By Lemma 2.6, there is a symmetric neighborhood V_0 in V_1 such that $x_1x_2 \in V_0$ for all $x_1,x_2 \in V_1$. Suppose W_1 is a neighborhood of the identity in C. By Lemma 2.6, there is a symmetric neighborhood W_0 in W_1, such that $c_1c_2 \in W_0$ for all $c_1,c_2 \in W_1$.

Suppose $E = C \underset{\theta}{\times} X$. By [3, Theorem 2.28], $C \underset{\theta}{\times} X$ is a topological local group with the product

$$(c_1,x_1)(c_2,x_2) = (c_1(\theta_{x_1}c_2)\eta(f(x_1,x_2)),x_1x_2)$$

for every $x_1,x_2 \in V_1$ and $c_1,c_2 \in W_1$;

where $\theta_{x_1} : W_0 \to C$ is a local automorphism.

The identity of E is $(1,1)$ and the inverse is given by

$(c,x)^{-1} = (\theta_x^{-1}(c^{-1}\eta(f(x,x^{-1}))^{-1}),x^{-1})$ for $x \in X$ and $c \in C$.

The map $\pi : E \to X$, $\pi : (c,x) \mapsto x$ is a strong homomorphism, since π is the projection of $C \underset{\theta}{\times} X$ onto X. It is clear that π is open and continuous. The kernel π = C_0 consists of the elements $(c,1)$ with the product $(c_1,1)(c_2,1) = (c_1c_2,1)$ whenever c_1c_2 is defined. Since f is normalized then $f(1,x) = 1$, for every $x \in X$. So C_0 can be identified with C by the correspondence $\eta : c \leftrightarrow (cf(1,1)^{-1},1)$.

Let $u(x) = (1,x)$. If $x_1,x_2 \in V_1$, we define

$$u(x_1)u(x_2) = (\eta(f(x_1,x_2)),1)u(x_1x_2) \quad (3.4)$$

We have

$$\begin{aligned} u(x_1)u(x_2)u(x_1x_2)^{-1} &= (1,x_1)(1,x_2)(1,x_1x_2)^{-1}; \\ &= (f(x_1,x_2),x_1x_2)(1,x_1x_2)^{-1}; \\ &= (f(x_1,x_2)f(1,x_1x_2)^{-1},1)(1,x_1x_2)(1,x_1x_2)^{-1}; \\ &= (f(x_1,x_2),1). \end{aligned}$$

We can verify that

$$u(x)cu(x)^{-1} = (\theta_x c,1) \quad (3.5)$$

$$\begin{aligned} u(x)cu(x)^{-1} &= (1,x)(c,1)(1,x)^{-1}; \\ &= (\theta_x cf(x,1),x)(1,x)^{-1}; \\ &= (\theta_x c,1)(1,x)(1,x)^{-1}; \\ &= (\theta_x c,1). \end{aligned}$$

Then, by (3.4) and (3.5), (f,θ) is a pair factor set of (E,π,η) with the local cross-section u.□

Remark 3.17*Let (f,θ) be a pair factor set of X to C. By Propositions 3.13, 3.16, 3.14, every pair factor set corresponds to an element of $H_L^2(X,C)$. Therefore, (f,θ) determines $\varepsilon \in Ext_{c_L}(X,C)$, $\varepsilon = (E,\pi,\eta)$. Now, let $\varepsilon' \in Ext_{c_L}(X,C)$, $\varepsilon' = (E',\pi',\eta')$ be another local extension and $\varepsilon' \equiv \varepsilon$.*

By Proposition 3.13, there is a pair factor set (f',θ') which corresponds to ε'. By Definition 3.7, choose an arbitrary open continuous strong isomorphism $\sigma : E;E'$ such that $\pi'\sigma = \pi$ and $\sigma\eta(c) = \eta'(c)$ for $c \in C$. Then, $u' = \sigma u$ is a continuous local cross-section of ε'. As in Proposition 3.14, (f',θ') corresponding to u' is identical with (f,θ) corresponding to u. Hence, equivalent local extensions of C by X determine the same element of $H_L^2(X,C)$.

Theorem 3.18 *Let X and C be topological local groups and C abelian. There is a one to one corresponding between the second cohomology of a topological local group $H_L^2(X,C)$ and the group of equivalence classes of topological local extensions with continuous local crossed-sections.*

*Proof.*By [4,Theorem 2.9], $H_L^2(X,C)$ is a group. Then, it is immediate by, Propositions 3.13 and 3.16.□

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