MV- Optimality of Nearest Neighbour Balanced Block Designs Using first order and Second Order Correlated Models.

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Research Article

Abstract: The block designs for observations correlated in one dimension and the universal optimality on Nearest Neighbor Balanced Block Designs (NNBD) using first order and second order correlated models (AR(1), MA(1) , ARMA (1,1) ,AR(2) and MA(2)) were investigated by SANTHARAM and PONNUSAMY (1996). In this paper we have investigated and addressed the MV-optimality Nearest Neighbor Balanced Block Designs (NNBD) using AR(1), MA(1) , ARMA(1,1) ,AR(2) and MA(2) models.

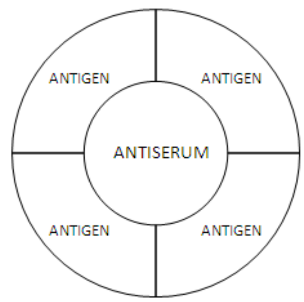
Key words: Auto-regressive model; Moving average model; ARMA model. Optimal experimental design: MV-optimality.

1. Introduction:

Serology is a branch of Biometrics which is concerned with the study of virus and viral preparations. Many studies concerned with viral preparation require the arrangement of antigens in a place so that each antigen has two other antigens as its neighbours. In analysis of such experiment the classical design may not perform efficiently. Therefore Rees, D. H. (1967) introduced neighbour structure. The following is the experiment considered by Rees, D. H. (1967) Nearest Neighbour Balanced Block Design. As seen in the figure1 the observations are available are correlated, therefore the usual assumption like independence of observation in the analysis of classical comparative experiments may not be valid. Therefore there is necessity for the use of Nearest Neighbour Balanced Block Design.

In biometrical science we can cite many areas where this kind of correlated structure exists. Now consider the viral preparation given above. Let there be t kinds of antigens to be arranged on b plates, each containing k antigens. Each antigen appear r times (but not necessarily on r different plates) and is a neighbour of every other antigen exactly λ times.

Rees used circular neighbouring block designs, where as in the present paper we are dealing with one dimensional block designs. Rees, D. H. (1967) used incomplete neighbour design $(k < t)$ in his experiment.



The parameters of the design are

$t = 9,\ b = 9,\ r = 4,\ \lambda = 1$ and the 9 plates are

$P_1 = (5,6,4,1),\ P_2 = (6,7,5,2),\ P_3 = (7,8,6,3)$

$P_4 = (8,9,7,4),\ P_5 = (9,1,8,5),\ P_6 = (1,2,9,6)$

$P_7 = (2,3,1,7),\ P_8 = (3,4,2,8),\ P_9 = (4,5,3,9)$

In the present paper we have taken complete NNBD $(k < v)$with the parametric structures.

$v = 3, \quad b = 3n \pm 1, \quad k = 5, \quad \lambda = 1$

and investigated the optimality of NNBD ($\rho_1 = 0.1, \rho_2 = 0.1, : \rho_2 = 0.2; \ldots \rho_1 = 0.9, \rho_2 = 0.9$ where $\rho_1$and ρ_2 are the correlation coefficients) when errors according to AR(1), MA(1) , ARMA(1,1) ,AR(2) and MA(2) models.

In the design of experiments, observations are assumed to be uncorrelated but correlated observations are unavoidable in practice. Observations are correlated either because of the nature and layout of the plots, some cumulative effects through time, pest infections from neighbouring plots, or because of some other local factors which blocking cannot remove. Experiments in agriculture, horticulture and forestry often show neighbour effects. Rees, D. H. (1967) introduced neighbour design in serology and defined it as a collection of circular blocks in which any two distinct treatments appear as neighbours equally often. UDDIN, N.,(2008) has constructed MV–Optimality of block design for 3 treatments in $b = 3n \pm 1$ blocks of each size 3 and under the assumption that the blocks behave independently but there is a correlation among the observations with the same block according to AR(1) Model.

1.1 Optimal Design

Optimal designs are experimental designs that are generated based on particular optimality criterion and are generally optimal only for a specific statistical model.

1.2 Optimality Criterion

An optimality criterion is used to study the goodness of a design. An optimality criterion is a single number that summarizes how good a design is, and it is maximized or minimized by an optimal design. Some known criteria are G, D, A, E and I – optimality. Recently developed optimality criteria are Universal, MS and MV –optimality. Here we have given the criteria of MV – optimality.

2. MV – Optimality

A design $d^* \in D$ is said to be MV – optimal iff

$$\max_{\substack{1\le i\le s \\ s+1\le j\le t}} \{\mathrm{Var}_{d^*}(\hat{T}_i - \hat{T}_j)\} \le \max_{\substack{1\le i\le s \\ s+1\le j\le t}} \{\mathrm{Var}_d(\hat{T}_i - \hat{T}_j)\}$$

Where D the class of all equireplicate is connected designs and d is any other competing design in D.

2. MV – OPTIMALITY OF NEARST NEIGHBOUR BLOCK DEISGN

UDDIN, N., (2008) constructed MV-Optimality of block design for 3 treatments in $b = 3n \pm 1$ blocks of each size 3 and under the assumption that the blocks behave independently but there is a correlation among the observations with the same block according to AR(1) model .In this connection we have constructed the blocks for AR(1), MA(1), ARMA(1,1) ,AR(2) and MA(2) models.

A block design d is defined as an allocation of v treatments to bk experimental units which are arranged into b blocks each having k units.

Example

Let d be a design

$$(v = 3, b = 3n \pm 1, k = 5\ \lambda = 1)\ \ for\ all\ n \ge 1;$$

Block:1 (1,2,3)
Block: 2 (2,3,1)
Block: 3 (3,1,2)

Assuming that the observations within the same block are correlated according to the second order autoregressive process and second order moving average process.

`We assume the following model

$$\mathrm{Y_d} = \mathrm{I_{3b}} + \mathrm{X_d}\tau + \mathrm{Z}\beta, \mathrm{Cov}(\epsilon) = \sigma^2 \sum \quad (1)$$

Where,

Y_d = block major order is the $3b \times 1$ column vector of observed response obtained from a design d,

$I_{3b} = 3b \times 1$ column vector of ones, $\tau = 3 \times 1$ vector of treatment effect,

$X_d = 3b \times 3$ plot-treatment design matrix, $\beta = 3 \times 1$vector of fixed block effects,

$Z = I_b \times I_3$ plot-block incident matrix.

If the errors within a block follow an second order Autoregressive model,

If the errors within a block follow an Autoregressive model, AR (1) then

$$\sum = \mathrm{I(b)} \otimes (1-\rho^2)^{-1}\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \text{for all } \rho \ge 0 \quad (2)$$

If the errors within a block follow a first order Moving Average model then MA (1) is as follows

$$\Sigma = \mathrm{I(b)} \otimes \begin{pmatrix} 1+\rho^2 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1+\rho^2 \end{pmatrix} \text{for all } \rho \ge 0 \quad (3)$$

If the errors within a block follow an Autoregressive Moving Average model, ARMA (1, 1) then

$$\sum = \mathrm{I(b)} \otimes \begin{pmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{pmatrix}, \text{for all } \rho \ge 0 \quad (4)$$

If the errors within a block follow an second order Autoregressive model, AR(2) then

$$\Sigma = I_b \otimes \begin{pmatrix} r_0 & r_1 & r_2 \\ r_1 & r_0 & r_1 \\ r_2 & r_1 & r_0 \end{pmatrix}, \ for\ all\ \rho_1, \rho_2 \ge 0 \quad (5)$$

Where,

$$r_0 = {(1-\rho_2)}\Big/{(1+\rho_2)\{(1-\rho_2)^2 - \rho_1^2\}},$$

$$r_1 = \{\rho_1^2 / 1-\rho_2\} r_0,$$

$$r_2 = \{[\rho_1^2 / 1-\rho_2] + \rho_2\} r_0,$$

If the error within a block follow a second order Moving Average model, MA(2) then

$$\Sigma = I_b \otimes \begin{pmatrix} 1+\rho_1^2+\rho_2^2 & \rho_1+\rho_1\rho_2 & \rho_2 \\ \rho_1+\rho_1\rho_2 & 1+\rho_1^2+\rho_2^2 & \rho_1+\rho_1\rho_2 \\ \rho_2 & \rho_1+\rho_1\rho_2 & 1+\rho_1^2+\rho_2^2 \end{pmatrix} \ for all \rho_1, \rho_2 \ge 0 \quad (6)$$

The least square information matrix

$$C_d = X_1' \Sigma^{-1} X_1 - X_1' \Sigma^{-1} X_2 (X_2' \Sigma^{-1} X_2)^{-1} X_2' \Sigma^{-1} X_1$$

Universally optimal design which includes MV-optimal design is often characterized using two sufficient conditions of Kiefer's(1975),

1. A design d^* such that C_d^* is completely symmetric.

2. trace $(C_d^*) \geq$ trace (C_d^*) for all d in D_b.

Uddin (2008b) gives MV–optimal designs in $D_{b=3n\pm1}$ under model (1)

$C_{dij}^* = (\mathrm{i,j})^{\mathrm{th}}$, $C_{dij}^* =$ element of C_d^*.

For any $d \in D_b$, the following inequality holds (Lee and Jacroux, 1987);

$$\mathrm{Var_d}(\hat{\tau}_i - \hat{\tau}_j) \geq \frac{C_{dii} + C_{djj} + 2C_{dij}}{C_{dii}\,C_{djj} - {C_{dij}}^2}$$

Variance of the generalized least squares estimates of treatment differences,

$$\mathrm{Var_d}(\hat{\tau}_1 - \hat{\tau}_2) = \frac{C_{d11} + C_{d22} + 2C_{d12}}{C_{d11}\,C_{d22} - {C_{d12}}^2}$$

$$\mathrm{Var_d}(\hat{\tau}_1 - \hat{\tau}_3) = \frac{C_{d11}}{C_{d11}\,C_{d22} - {C_{d12}}^2}$$

$$\mathrm{Var_d}(\hat{\tau}_2 - \hat{\tau}_3) = \frac{C_{d22}}{C_{d11}\,C_{d22} - {C_{d12}}^2}$$

From Uddin (2008b) $d_1^* \in D_{b=3n+1}$ to denote the design having *n* copies of the blocks (1,2,3); (3,1,2) and *(n+1)* copies of the block (1,3,2).

Also $d_2^* \in D_{b=3n-1}$ to denote the design having *n* copies of the blocks (1,2,3); (3,1,2) and *(n-1)* block (1,3,2)

4.1 MV- Optimality Of Nearest Neighbour Balanced Block Designs Using first and Second Order Correlated Models

For $AR(1), MA(1)$ *and* $ARMA(1,1)$

Case 1: $b = 3n + 1, n = 2;\ (v = 3, b = 7, k = 3)$

The variance of the estimates of the treatment differences for D_1^* & D_1 is shown in Table 4.1.

Case 2: $b = 3n - 1, n = 2;\ (v = 3, b = 7, k = 3)$

The variance of the estimates of the treatment differences for D_2^* & D_2 is shown in Table 4.2.

For AR(2) and MA(2),

Case 1: $b = 3n + 1, n = 2;\ (v = 3, b = 7, k = 3)$

The variance of the estimates of the treatment differences for D_1^* & D_1 is shown in Table 4.3.

Case 2: $b = 3n - 1, n = 2;\ (v = 3, b = 7, k = 3)$

The variance of the estimates of the treatment differences for D_2^* & D_2 is shown in Table 4.4.

Conclusion

From Table 4.1 we conclude that

- the variance of the estimates of the treatment differences for D_1^* is less than D_1 for $\rho = 0.1$, 0.2, 0.3,…..,0.9 under AR(1), MA(1) and ARMA(1,1) model,
- so we conclude that the design D_1^* is MV-optimal comparing with D_1.

From Table 4.2 we conclude that

- the variance of the estimates of the treatment differences for D_2^* is less than D_2 for $\rho= 0.1$, 0.2, 0.3,…..,0.9 under AR(1), MA(1) and ARMA(1,1) model,
- so we conclude that the design D_2^* is MV-optimal comparing with D_2.

From table 4.3 we conclude that

- the variance of the estimate of the treatment differences for D_1^* is less than D_1 for ρ_1, ρ_2 = *(0.1,0.1), (0.2,0.2),…(0.9,0.9)* under AR(2) and MA(2) model,
- so we conclude that design D_1^* is MV – optimal comparing with D_1.

From table 4.4 we conclude that

- the variance of the estimate of the treatment differences for D_2^* is less than D_2 for ρ_1, ρ_2 = *(0.1,0.1), (0.2,0.2),…(0.9,0.9)* under AR(2) and MA(2) model,
- so we conclude that design D_2^* is MV – optimal comparing with D_2.

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APPENDIX

Table 4.1 – AR(1), MA(1) and ARMA(1) when $b = 3n + 1$

Error model		ρ = 0.1	ρ = 0.2	ρ = 0.3	ρ = 0.4	ρ = 0.5	ρ = 0.6	ρ = 0.7	ρ = 0.8	ρ = 0.9
AR(1)	D_1^*	0.4814	0.4598	0.4365	0.4128	0.3903	0.3701	0.3537	0.342	0.3353
	D_1	1.6246	1.3	1.0228	0.7837	0.6391	0.3073	0.4964	5.0474	2.3377
MA (1)	D_1^*	0.4815	0.4583	0.4286	0.3889	0.3333	0.25	0.1111	0.1667	0.2662
	D_1	1.5853	1.159	0.7477	0.3904	1.235	1.0246	2.8646	4.0883	5.4064
ARMA (1,1)	D_1^*	0.4779	0.4685	0.4508	0.3876	0.1982	0.3237	1.8435	7.3934	42.285
	D_1	1.2729	0.6664	0.4738	0.5037	2.8564	1.6755	5.0185	20.342	128.973

Table 4.2 – AR(1), MA(1) and ARMA(1) when $b = 3n - 1$

Error model		ρ = 0.1	ρ = 0.2	ρ = 0.3	ρ = 0.4	ρ = 0.5	ρ = 0.6	ρ = 0.7	ρ = 0.8	ρ = 0.9
AR(1)	D_2^*	0.3256	0.3293	0.3279	0.3216	0.3107	0.2962	0.2790	0.2603	0.2411
	D_2	0.3985	0.3923	0.3816	0.3731	0.3625	0.3539	0.3500	0.3539	0.3691
MA(1)	D_2^*	0.3253	0.3175	0.3253	0.3140	0.2883	0.2305	0.0453	0.3889	0.2247
	D_2	0..4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000	0.4000
ARMA(1,1)	D_2^*	0.3415	0.3714	0.3941	0.3748	0.0132	0.4624	0.5086	1.8971	2.9781
	D_2	0.4162	0.4666	0.5573	0.7007	0.9208	1.2689	1.8699	3.1134	6.9865

Table 4.3 – AR(2), and MA(2) when $b = 3n + 1$

Error model		ρ_1=0.1 ρ_2=0.1	ρ_1=0.2 ρ_2=0.2	ρ_1=0.3 ρ_2=0.3	ρ_1=0.4 ρ_2=0.4	ρ_1=0.5 ρ_2=0.5	ρ_1=0.6 ρ=0.6	ρ_1=0.7 ρ_2=0.7	ρ_1=0.8 ρ_2=0.8	ρ_1=0.9 ρ=0.9
AR(2)	D_1^*	0.1080	0.0608	0.0278	0.0076	0.0042	0.0008	0.0090	0.0291	0.0546
	D_1	0.1170	0.0727	0.0385	0.0142	0.0077	0.0012	0.0127	0.0341	0.0580
MA(2)	D_1^*	0.1569	0.1478	0.1401	0.1438	0.1476	0.1544	0.1637	0.1753	0.1893
	D_1	0.1578	0.1512	0.2615	0.1585	0.1714	0.1890	0.2100	0.2332	0.2584

Table 4.4 – AR(2), and MA(2) when $b = 3n - 1$

Error model		ρ_1=0.1 ρ_2=0.1	ρ_1=0.2 ρ_2=0.2	ρ_1=0.3 ρ_2=0.3	ρ_1=0.4 ρ_2=0.4	ρ_1=0.5 ρ_2=0.5	ρ_1=0.6 ρ=0.6	ρ_1=0.7 ρ_2=0.7	ρ_1=0.8 ρ_2=0.8	ρ_1=0.9 ρ=0.9
AR(2)	D_2	0.1548	0.0896	0.0425	0.0124	0.00655	0.0007	0.0120	0.0398	0.0758
	D_2	0.1638	0.1018	0.0539	0.0200	001083	0.0016	0.0178	0.0478	0.0812
MA(2)	D_2	0.2196	0.2062	0.1995	0.1985	0.2023	0.2101	0.2211	0.2354	0.2530
	D_2	0.2209	0.2117	0.2123	0.2220	0.3000	0.2646	0.2940	0.3266	0.3618