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On Vanishing of Pure Extensions of Torsion-Free Locally Compact Abelian Groups

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Research Article

Abstract: Let ℓ be the class of all torsion-free, locally compact abelian (LCA) groups. In this paper, we determine the LCA groups G such that every pure extension $0 \to X \to Y \to G \to 0$ splits for each X in ℓ.

keywords: Pure extension, split, torsion-free,locally compact abelian group.

MSC 2000: Primary 22B05, Secondary 22A05

1.Introduction

Throughout, all groups are Hausdorff abelian topological groups and will be written additively. Let £ denote the categ $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in £ is said to be proper exact if ϕ and ψ are proper morphisms. In this case, the sequence is called an extension of A by C. A subgroup H of a group C is called pure if $nH = H \cap nC$ for all positive integers n. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a pure extension if $\phi(A)$ is pure in B. Following Fulp and Griffith [5], we let $Ext(C,A)$ denote the (discrete) group of extensions of A by C. A subgroup H of a group G is called pure if $nH = H \cap nG$ for all positive integers n. The elements represented by pure extensions of A by C form a subgroup of $Ext(C,A)$ which is deenoted by $Pext(C,A)$. Assume that $\Im$ is any subcategory of £. This paper is part of an investigation which answers the following question: under what conditions on G, $Pext(X,G)=0$ or $Pext(G,X)=0$ for all $X \in \Im$?. Some answers are given in [2],[7] in the category of locally compact abelian groups. Let ℓ be the category of all torsion-free, locally compact abelian groups. In [3], Fulp has determined the LCA groups G such that $Pext(X,G)=0$ for all $X \in \ell$. In this paper, we determine the discrete or compact groups G satisfying $Pext(G,X)=0$ for all $X \in \ell$.

The additive topological group of real numbers is denoted by R , Q is the group of rationals , Z is the group of integers and $Z(n)$ is the cyclic group of order n. By G_d we mean the group G with discrete topology and tG is the torsion part of G. The topological isomorphism will be denoted by $\cong$. For more on locally compact abelian groups, see [4].

2.Splitting in the category of torsion-free, locally compact abelian groups

In this section, we determine the structure of a discrete or a compact group G such that $Pext(G,X)=0$ for all torsion-free groups $X \in$ £.

Recall that a discrete group A is said to be cotorsion if for any discrete torsion-free group B , Ext(B,A)=0.

Lemma 2.1 $Pext(Q/Z, X)=0$ for all torsion-free groups $X \in$ £.

Proof. Let X be a discrete torsion-free group. By [1, Theorem 52.3], $Ext(Q/Z, X) \cong Hom(Q/Z, D/X)$ where D is a divisible hull of X. By [1, Theorem 46.1 and Theorem 54.6], $Hom(Q/Z, D/X)$ and so $Ext(Q/Z, X)$ is a reduced,algeraically compact and cotorsion group. It follows from [1, Prposition 54.2] that

$$Pext(Q/Z, X) = \bigcap_{n=1} nExt(Q/Z, X) = 0$$

Now suppose that X is a torsion-free group in £. Then $Pext(Q/Z, X) \cong Pext(Q/Z, X_d) = 0$.□

Theorem 2.2 *Let* A *be a discrete group. Then* $Pext(A,X)=0$ *for all torsion-free groups* $X \in$ £ if and only if $A \cong (\oplus_{\sigma} Z) \oplus B$ where B is a discrete torsion group.

Proof. Let X be a torsion-free group. By [2, Proposition 4], we have the following exact sequence:

$$(1) \ldots \to Hom(tA, X) \to Pext(A/tA, X) \to Pext(A, X) \to \ldots$$

Since tA is torsion and X a torsion-free group, so $Hom(tA, X) = 0.$ Now assume that $Pext(A, X) = 0$. *By (1),* $0 = Pext(A/tA, X) = Ext(A/tA, X)$. *It follows from [3, Proposition 2] that* $A/tA \cong \oplus_{\sigma} Z$. *By [1, Theorem 28.2],* $A \cong (\oplus_{\sigma} Z) \oplus B$.

Conversely, suppose that $A \cong (\oplus_{\sigma} Z) \oplus B$ where B is discrete torsion . If X is torsion-free, then

$$Pext(A, X) \cong Pext((\oplus_{\sigma} Z), X) \oplus Pext(B, X)$$

It is enough to show that $Pext(B, X) = 0$. There exists a pure exact sequence

$$0 \to F \to B \to D \to 0$$

where F is a direct sum of finite cyclic groups and, D, a divisible torsion groupS. Consider the exact sequence

$$\ldots \to Pext(D, X) \to Pext(B, X) \to Pext(F, X) \to \ldots$$

Since $Pext(D, X) \cong \oplus Pext(Q/Z, X_d)$, so by lemma 2.1, $Pext(D, X) = 0$. Clearly $Pext(F, X) = 0$. So $Pext(B, X) = 0$.□

Lemma 2.3 Let X and Y be two topological spaces, $A \subseteq X$ *and* $f : X \to Y$ *an open map. If* $f^{-1}(f(A)) = A$, *then* $f : A \to f(A)$ *is open.*

Proof. It is clear.□

Theorem 2.4 Let X be a pure, closed and torsion-free subgroup of G. Then $G[n] \cong (G/X)[n]$ for all n.

Proof. Let $\rho : G \to G/X$ be the natural map. For a fixed n, we show that $\rho|G[n] : G[n] \to (G/X)[n]$ is a topological isomorphism. Assume that $g + X \in (G/X)[n]$. Then $ng \in X$. Since X is pure, $ng = nx$ for some $x \in X$. So $n(g - x) = 0$ for some $x \in X$. Therefore, $g - x \in G[n]$ and $\rho(g - x) = g + X$,that is $\rho|G[n]$ is surjective. Now suppose that $\rho(g) = 0$ for some $g \in G[n]$. Then $g \in X$ and $ng = 0$. Since X is torsion-free, $g = 0$. So $\rho|G[n]$ is injective. Since ρ is open and $\rho|G[n]$ is injective, so by lemma 2.3, $\rho|G[n]$ is open. Hence $G[n] \cong (G/X)[n]$.□

Definition 2.5 Let $E : 0 \to X \to K \to G \to 0$ be an extension in £. Then E is said to be $*-$pure if the sequence

$$0 \to X[n] \to K[n] \to G[n] \to 0$$

is proper exact for all n. Following Loth [6], we let $Pext(G, X)^*$ denote the group of $*-$ pure extensions of X by G.

Lemma 2.6 Let $G \in$ £. Then

$$Pext(G, X) \subseteq Pext(G, X)^*$$

for all torsion-free groups $X \in$ £.

Proof. Let $E : 0 \to X \xrightarrow{i} K \to G \to 0$ be a pure extension of torsion-free group X by G. We may assume that i is the inclusion homomorphism. So X is a pure, closed and torsion-free subgroup of K. Now by Theorem 2.4, $K[n] \cong G[n]$ for all n. Since $X[n] = 0$ so $0 \to X[n] \to K[n] \to G[n] \to 0$ is proper exact for all n.

Theorem 2.7 Let G be a compact group. Then $Pext(G, X) = 0$ for all torsion-free groups X if and only if G is torsion.

Proof. Assume that G is a compact group such that $Pext(G, X) = 0$ for each torsion-free group X. By [7, Lemma 2.13], $Pext(\hat{Q}, \hat{G}) \cong Pext(G, Q) = 0$. So $Ext(\hat{Q}, \hat{G}) = 0$. Since the sequence

$$\cdots \to 0 = Ext(\hat{Q}, \hat{G}) \to Ext(\hat{Q}, \hat{G}/t\hat{G}) \to 0$$

Is exact, $Ext(\hat{Q}, \hat{G}/t\hat{G}) = 0$. Now consider the exact sequence

$$0 = Hom((Q/Z\hat{)}, \hat{G}/t\hat{G}) \to Ext(\hat{Z}, \hat{G}/t\hat{G}) \to Ext(\hat{Q}, \hat{G}/t\hat{G}) = 0$$

So $\hat{G}/t\hat{G}$ is isomorphic to $Ext(\hat{Z},\hat{G}/t\hat{G})=0$ and therefore $\hat{G}=t\hat{G}$. It follows that $\hat{G}$ is torsion. Since $Pext(Q,\hat{G})\cong Pext(G,\hat{Q})=0$, so $Ext(Q,\hat{G})=0$. Hence $\hat{G}$ is cotorsion. By [1, Corollary 54.4], $\hat{G}\cong B\oplus D$ where B is a bounded group and D a divisible group. So $G\cong\hat{B}\oplus\hat{D}$. Clearly, $Ext(\hat{D},X)=0$ for each torsion-free group X. Consequently by [3, Theorem 2], $\hat{D}=0$ or $D=0$. Conversely, let G be a compact torsion group. Then by [6, Theorem 4.2] and Lemma 2.6, $Pext(G,X)=0$ for all torsion-free groups X.

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