International Journal of Statistika and Mathematika, ISSN: 2277- 2790 E-ISSN: 2249-8605, Volume 6, Issue 3, 2013 pp 115-120

Maximum Modulus and Maximum Terms-Related Growth Properties of Entire Functions Based on Relative Type and Relative Weak Type

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Research Article

Abstract: In the paper we study the comparative growth properties of composite entire function on the basis of relative order, relative type and relative weak type with respect to another entire function.

Keywords and phrases: Entire function, maximum term, maximum modulus, composition, growth, relative order, relative type, relative weak type.

AMS Subject Classification (2010) : 30D35, 30D30, 30D20.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers . Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\boldsymbol{\mu(r,f)}$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\boldsymbol{\mu(r,f)} = \max_{n=0} (|\boldsymbol{a_n}|\boldsymbol{r^n})$ and the maximum modulus $\boldsymbol{M(r,f)}$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\boldsymbol{M(r,f)} = \max_{|z|=r} |f(z)|$ **.** In the sequel we use the following notation **:**

$$\log^{[k]}\boldsymbol{x} = \log(\log^{[k-1]}\boldsymbol{x}) \text{ for } \boldsymbol{k = 1, 2, 3}, \ldots \text{ and } \log^{[0]}\boldsymbol{x} = \boldsymbol{x}.$$

To start our paper we just recall the following definition:

Definition 1 The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \textit{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Using the inequalities $\boldsymbol{\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)}$ { *cf.* [8] }, for $\boldsymbol{0 \leq r < R}$ one may verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r} \quad \textit{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log r}.$$

Definition 2 The type σ_f of an entire function f is defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}, \ 0 < \rho_f < \infty.$$

Datta and Jha [3] introduced the definition of weak type of a meromorphic function of finite positive lower order in the following way :

Definition 3 [3] The weak type τ_f of an entire function f of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}}.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse $M_g^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_g^{-1}(s) = \infty$.

Bernal [1] introduced the definition of relative order of an entire function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\rho_g(f) = \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\} = \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

The definition coincides with the classical one [9] if $g(z) = \exp z$.

Similarly one can define the relative lower order of an entire function f with respect to an entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Datta and Maji [4] gave an alternative definition of relative order and relative lower order of an entire with respect to another entire in the following way :

Definition 4 [4] The relative order $\rho_g(f)$ and relative lower order $\lambda_g(f)$ of an entire function f with respect to an entire function g are defined as follows:

$$\rho_g(f) = \limsup_{r \to \infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r \to \infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log r}.$$

Recently Roy [5] introduced the notion of relative type of two entire functions in the following manner :

Definition 5 [5] Let f and g be any two entire functions such that $0 < \rho_g(f) < \infty$. Then the relative type $\sigma_g(f)$ of f with respect to g is defined as:

$$\sigma_g(f) = \inf \{ k > 0 : M_f(r) < M_g(kr^{\rho_g(f)}) \text{ for all sufficiently large values of } r\} = \limsup_{r \to \infty} \frac{M_g^{-1} M_f(r)}{r^{\rho_g(f)}}.$$

Analogously to determine the relative growth of two entire functions having same non zero finite relative lower order with respect to another entire function, one may introduce the definition of relative weak type (in the notion of Datta and Jha [3]) of an entire function f with respect to another entire function g of finite positive relative lower order $\lambda_g(f)$ in the following way :

Definition 6 The relative weak type $\tau_g(f)$ of an entire function f with respect to another entire function g having finite positive relative lower order $\lambda_g(f)$ is defined as:

$$\tau_g(f) = \liminf_{r \to \infty} \frac{M_g^{-1} M_f(r)}{r^{\lambda_g(f)}}.$$

Considering $g = \exp z$ one may easily verify that Definition 5 and Definition 6 coincide with the classical Definition 2 and Definition 3 respectively.

In the paper we study some relative growth properties of maximum term and maximum modulus of composition of entire functions with respect to another entire function on the basis of relative order, relative type and relative weak type. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [10].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [2] If f and g are two entire functions then for all sufficiently large values of r,

$$M_{fog}(r) \le M_f(M_g(r)).$$

Lemma 2 [7] Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,

$$\mu_{fog}(r) \le \frac{\alpha}{\alpha-1} \mu_f\left(\frac{\alpha R}{R-r} \mu_g(R)\right).$$

Lemma 3 [1] Suppose f is an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,

$$M_f(\alpha r) \ge \beta M_f(r).$$

Lemma 4 [4] If f is entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,

$$\mu_f(\alpha r) \ge \beta \mu_f(r).$$

Lemma 5 Let f and g be any two entire functions. Then for any $\alpha > 1$,

$$(i)\ M_h^{-1} M_f(r) \le \mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)} \mu_f(\alpha r)\right] \text{ and } (ii)\ \mu_h^{-1}\mu_f(r) \le \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)} M_f(r)\right].$$

Proof. Taking $R = \alpha r$ in the inequalities $\boldsymbol{\mu_h(r)} \le M_h(r) \le \frac{\boldsymbol{R}}{\boldsymbol{R-r}} \mu_h(R)$ { *cf.* [8] }, for $\boldsymbol{0 \le r < R}$ we obtain that

$$M_h^{-1}(r) \le \mu_h^{-1}(r)$$

$$\text{and } \mu_h^{-1}(r) \le \alpha M_h^{-1}\left(\frac{\alpha r}{(\alpha-1)}\right).$$

Since $M_h^{-1}(r)$ and $\mu_h^{-1}(r)$ are increasing functions of r, the lemma follows from the above and the inequalities $\boldsymbol{\mu_f(r)} \le M_f(r) \le \frac{\boldsymbol{\alpha}}{\boldsymbol{\alpha-1}} \mu_f(\alpha r)$ { *cf*. [8] }.

2. Theorems

In this section we present the main results of the paper.

Theorem 1 Let f, g and h be any three entire functions such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$ and $\sigma_g < \infty$. Then for any $\beta > 1$

$$\limsup_{r \to \infty} \frac{\log \mu_h^{-1} \mu_{fog}(r)}{\log \mu_h^{-1} \mu_f (\exp(\beta r)^{\rho_g})} \le \frac{\sigma_g \cdot \rho_h(f)}{\lambda_h(f)}.$$

Proof. Taking $R = \beta r$ in Lemma 2 and in view of Lemma 4 we have for all sufficiently large values of r that

$$\mu_{fog}(r) \le \left(\frac{\alpha}{\alpha-1}\right) \mu_f\left(\frac{\alpha\beta}{(\beta-1)} \mu_g(\beta r)\right)$$

$$i.e.,\ \mu_{fog}(r) \le \mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)} \mu_g(\beta r)\right).$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from above for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{fog}(r) \le \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)} \mu_g(\beta r)\right)$$

$$i.e.,\ \log \mu_h^{-1}\mu_{fog}(r) \le \log \mu_h^{-1}\mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)} \mu_g(\beta r)\right) \tag{1}$$

$$i.e., \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})} \le \frac{\log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})}$$

$$= \frac{\log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log\left\{\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right\}} \cdot$$

$$\frac{\log \mu_g\,(\beta r)+O(1)}{(\beta r)^{\rho g}} \cdot \frac{\log\{\exp(\beta r)^{\rho g}\}}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})} \tag{2}$$

$$i.e., \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})}$$

$$\le \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right)}{\log\left\{\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r)\right\}} \cdot$$

$$\limsup_{r\to\infty} \frac{\log \mu_g\,(\beta r)+O(1)}{(\beta r)^{\rho g}} \cdot \limsup_{r\to\infty} \frac{\log\{\exp(\beta r)^{\rho g}\}}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})} \cdot$$

$$i.e., \limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\rho g})} \le \rho_h(f)\cdot\sigma_g\cdot\frac{1}{\lambda_h(f)}\cdot$$

Thus the theorem is established.

In the line of Theorem 1 the following theorem can be proved :

Theorem 2 Let f , g and h be any three entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$ and $\sigma_g < \infty$. Then for any $\beta > 1$,

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_g\,(\exp(\beta r)^{\rho g})} \le \frac{\sigma_g\,.\,\rho_h(f)}{\lambda_h(g)}\;.$$

The proof is omitted.

With the help of Lemma 1 and in the line of Theorem 1 and Theorem 2 the following two theorems may be proved :

Theorem 3 Let f , g and h be any three entire functions such that $0 < \lambda_h(f) \le \rho_h(f) < \infty$ and $\sigma_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{\log M_h^{-1}M_f\,(\exp(r)^{\rho g})} \le \frac{\sigma_g\,.\rho_h(f)}{\lambda_h(f)}\;.$$

Theorem 4 Let f , g and h be any three entire functions with $\lambda_h(g) > 0$, $\rho_h(f) < \infty$ and $\sigma_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{\log M_h^{-1}M_g\,(\exp(r)^{\rho g})} \le \frac{\sigma_g\,.\;\rho_h(f)}{\lambda_h(g)}\;.$$

Using the notion of weak type, we may state the following two theorems without proof because it can be carried out in the line of Theorem 1 and Theorem 3 respectively.

Theorem 5 Let f , g and h be any three entire functions such that $0 < \lambda_h(f) = \rho_h(f) < \infty$ and $\tau_g < \infty$. Then for any $\beta > 1$,

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\log \mu_h^{-1}\mu_f\,(\exp(\beta r)^{\lambda g})} \le \tau_g\;.$$

Theorem 6 Let f , g and h be any three entire functions with $0 < \lambda_h(f) = \rho_h(f) < \infty$ and $\tau_g < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{\log M_h^{-1}M_f\,(\exp(r)^{\lambda g})} \le \tau_g\;.$$

Theorem 7 Let f , g and h be any three entire functions such that $(i)\; 0 < \rho_h(f) < \infty$, $(ii)\; \rho_h(f) = \rho_g$, $(iii)\; \sigma_g < \infty$, and $(iv)\; 0 < \sigma_h(f) < \infty$. Then for any α , $\beta > 1$ and $\gamma > 0$,

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f\,(r)} \le \left[\frac{(\alpha+\gamma\alpha-\gamma)\alpha\beta}{(\alpha-1)}\right]^{\rho_h(f)} \cdot \frac{\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}\;.$$

Proof. From (1) and the inequality $\mu(r,f) \le M(r,f)$ $\{cf.\,[8]\}$, we get for all sufficiently large values of r that

$$\log \mu_h^{-1}\mu_{fog}(r) \le (\rho_h(f)+\varepsilon)\{\log \mu_g(\beta r)+O(1)\}$$

$$i.e., \log \mu_h^{-1}\mu_{fog}(r) \le (\rho_h(f)+\varepsilon)\{\log M_g(\beta r)+O(1)\}\cdot \tag{3}$$

Using the definition of type we obtain from (3) for all sufficiently large values of r that

$$\log \mu_h^{-1}\mu_{fog}(r) \le (\rho_h(f)+\varepsilon)(\sigma_g+\varepsilon)\{\beta r\}^{\rho_g}+O(1)\cdot \tag{4}$$

Now in view of condition (ii) we obtain from (4) for all sufficiently large values of r that

$$\log \mu_h^{-1}\mu_{fog}(r) \le (\rho_h(f)+\varepsilon)(\sigma_g+\varepsilon)\{\beta r\}^{\rho_h(f)}+O(1) \tag{5}$$

Again in view of Lemma 4, Lemma 5 and the definition of relative type we get for a sequence of values of r tending to infinity that

$$\mu_h^{-1}\left[\frac{\alpha}{(\alpha-1)}\mu_f(\alpha r)\right] \ge M_h^{-1}M_f(r)$$

$$i.e.,\; \mu_h^{-1}\left[\mu_f\left(\frac{(\alpha+\gamma\alpha-\gamma)\alpha r}{(\alpha-1)}\right)\right] \ge M_h^{-1}M_f(r)$$

$$i.e.,\; \mu_h^{-1}\mu_f(r) \ge M_h^{-1}M_f\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}r\right)$$

$$i.e.,\; \mu_h^{-1}\mu_f(r) \ge (\sigma_h(f)-\varepsilon)\left\{\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}r\right\}^{\rho_h(f)} \tag{6}$$

Now from (5) and (6) it follows for a sequence of values r tending to infinity that

$$\frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \le \frac{(\rho_h(f)+\varepsilon)(\sigma_g+\varepsilon)\{\beta r\}^{\rho_h(f)}+O(1)}{(\sigma_h(f)-\varepsilon)\left\{\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}r\right\}^{\rho_h(f)}}\cdot$$

Since $\varepsilon > 0$ is arbitrary , it follows from above that

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \le \left[\frac{(\alpha+\gamma\alpha-\gamma)\alpha\beta}{(\alpha-1)}\right]^{\rho_h(f)} \frac{\rho_h(f).\;\sigma_g}{\sigma_h(f)}\cdot$$

Using the notion of weak type and relative weak type, we may state the following theorem without proof as it can be carried out in the line of Theorem 7:

Theorem 8 Let f , g and h be any three entire functions such that $(i)\; 0 < \rho_h(f) < \infty$, $(ii)\; \lambda_h(f) = \lambda_g$, $(iii)\; \tau_g < \infty$, and $(iv)\; 0 < \tau_h(f) < \infty$. Then for any α , $\beta > 1$ and $\gamma > 0$,

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f\,(r)} \le \left[\frac{(\alpha+\gamma\alpha-\gamma)\alpha\beta}{(\alpha-1)}\right]^{\lambda_h(f)} \cdot \frac{\rho_h(f)\cdot\tau_g}{\tau_h(f)}\;.$$

Similarly using the notion of type and relative weak type one may state the following two theorems without proof because those can also be carried out in the line of Theorem 7:

Theorem 9 Let f , g and h be any three entire functions such that $(i)\; \lambda_h(f) = \rho_g$, $(ii)\; \sigma_g < \infty$, and $(iii)\; 0 < \tau_h(f) < \infty$. Then for any α , $\beta > 1$ and $\gamma > 0$,

$$\liminf_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f\,(r)} \le \left[\frac{(\alpha+\gamma\alpha-\gamma)\alpha\beta}{(\alpha-1)}\right]^{\lambda_h(f)} \cdot \frac{\lambda_h(f)\cdot\sigma_g}{\tau_h(f)}\;.$$

Theorem 10 Let f , g and h be any three entire functions such that $(i)\; 0 < \rho_h(f) < \infty$, $(ii)\; \lambda_h(f) = \rho_g$, $(iii)\; \sigma_g < \infty$, and $(iv)\; 0 < \tau_h(f) < \infty$. Then for any α , $\beta > 1$ and $\gamma > 0$,

$$\limsup_{r\to\infty} \frac{\log \mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \leq \left[\frac{(\alpha+\gamma\alpha-\gamma)\alpha\beta}{(\alpha-1)}\right]^{\lambda_h(f)} \cdot \frac{\rho_h(f)\cdot\sigma_g}{\tau_h(f)}.$$

Theorem 11 Let f, g and h be any three entire functions with $(i)\ 0<\rho_h(f)<\infty$, $(ii)\ \rho_h(f)=\rho_g$, $(iii)\ \sigma_g<\infty$, and $(iv)\ 0<\sigma_h(f)<\infty$. Then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \leq \frac{\rho_h(f)\cdot\sigma_g}{\sigma_h(f)}.$$

Theorem 12 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h(f)<\infty$, $(ii)\ \lambda_h(f)=\lambda_g$, $(iii)\ \tau_g<\infty$, and $(iv)\ 0<\tau_h(f)<\infty$. Then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \leq \frac{\rho_h(f)\cdot\tau_g}{\tau_h(f)}.$$

Theorem 13 Let f, g and h be any three entire functions such that $(i)\ \lambda_h(f)=\rho_g$, $(ii)\ \sigma_g<\infty$, and $(iii)\ 0<\tau_h(f)<\infty$. Then

$$\liminf_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \leq \frac{\lambda_h(f)\cdot\sigma_g}{\tau_h(f)}.$$

Theorem 14 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h(f)<\infty$, $(ii)\ \lambda_h(f)=\rho_g$, $(iii)\ \sigma_g<\infty$, and $(iv)\ 0<\tau_h(f)<\infty$. Then

$$\limsup_{r\to\infty} \frac{\log M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \leq \frac{\rho_h(f)\cdot\sigma_g}{\tau_h(f)}.$$

The proof of Theorem 11, Theorem 12, Theorem 13 and Theorem 14 are omitted as those can be carried out in view of Lemma 1 and in the line of Theorem 7, Theorem 8, Theorem 9 and Theorem 10 respectively.

Theorem 15 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h(f)<\infty$, $(ii)\ 0<\sigma_h(f)<\infty$, $(iii)\ \rho_h(fog)=\rho_h(f)$, and $(iv)\ \sigma_h(fog)<\infty$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}}{(\alpha-1)^{2\rho_h(f)}} \cdot \frac{\sigma_h(fog)}{\sigma_h(f)}$$

and

$$\frac{(\alpha-1)^{2\rho_h(f)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}} \cdot \frac{\sigma_h(fog)}{\sigma_h(f)} \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}.$$

Proof . From the definition of relative type and in view of Lemma 3 and Lemma 5 we obtain for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{fog}(r) \leq \alpha M_h^{-1}\left[\frac{\alpha}{(\alpha-1)}M_{fog}(r)\right]$$
$$\leq \alpha M_h^{-1}\left[M_{fog}\left(\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right)\right]$$
$$i.e., \mu_h^{-1}\mu_{fog}(r) \leq \alpha(\sigma_h(fog)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(fog)} \tag{7}$$

and

$$\mu_h^{-1}\mu_f(r) \leq \alpha(\sigma_h(f)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(f)} \tag{8}$$

Also in view of Lemma 4 and Lemma 5 we obtain for a sequence of values of r tending to infinity that

$$\mu_h^{-1}\mu_{fog}(r) \geq M_h^{-1}M_{fog}\left(\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)r\right)$$
$$i.e.,\ \mu_h^{-1}\mu_{fog}(r) \geq (\sigma_h(fog)-\varepsilon)\left\{\left(\frac{(\alpha-1)r}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)\right\}^{\rho_h(fog)}$$

$$i.e.,\ \mu_h^{-1}\mu_{fog}(r) \geq \left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\rho_h(fog)} \cdot (\sigma_h(fog)-\varepsilon)\cdot r^{\rho_h(fog)} \tag{9}$$

and

$$\mu_h^{-1}\mu_f(r) \geq \left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\,\alpha}\right)^{\rho_h(f)} \cdot (\sigma_h(f)-\varepsilon)\cdot r^{\rho_h(f)}. \tag{10}$$

Now from (7) and (10) it follows for a sequence of values of r tending to infinity that

$$\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\,(\sigma_h(fog)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(fog)}}{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\,\alpha}\right)^{\rho_h(f)}\cdot(\sigma_h(f)-\varepsilon)\cdot r^{\rho_h(f)}}. \tag{11}$$

In view of the condition (*iii*) we get from (11) that

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{\alpha\,(\sigma_h(fog)+\varepsilon)\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)^{\rho_h(f)}}{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\,\alpha}\right)^{\rho_h(f)}\cdot(\sigma_h(f)-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \leq \frac{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}}{(\alpha-1)^{2\rho_h(f)}} \cdot \frac{\sigma_h(fog)}{\sigma_h(f)}. \tag{12}$$

Again from (8) and (9) we get for a sequence of values of r tending to infinity that

$$\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\,\alpha}\right)^{\rho_h(fog)}\cdot(\sigma_h(fog)-\varepsilon)\cdot r^{\rho_h(fog)}}{\alpha\,(\sigma_h(f)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(f)}}. \tag{13}$$

Since $\rho_h(fog)=\rho_h(f)$, we obtain from (13) that

$$\limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\,\alpha}\right)^{\rho_h(f)}}{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)^{\rho_h(f)}} \cdot \frac{(\sigma_h(fog)-\varepsilon)}{\alpha\,(\sigma_h(f)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)} \geq \frac{(\alpha-1)^{2\rho_h(f)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}} \cdot \frac{\sigma_h(fog)}{\sigma_h(f)}. \tag{14}$$

Thus the theorem follows from (12) and (14) .

In the line of Theorem 15 we may state the following theorem without proof :

Theorem 16 Let f, g and h be any three entire functions with $(i)\ 0<\rho_h(g)<\infty$, $(ii)\ 0<\sigma_h(g)<\infty$, $(iii)\ \rho_h(fog)=\rho_h(g)$ and $(iv)\ \sigma_h(fog)<\infty$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)} \leq \frac{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(g)}\cdot\alpha^{\rho_h(g)+1}}{(\alpha-1)^{2\rho_h(g)}} \cdot \frac{\sigma_h(fog)}{\sigma_h(g)}$$

and

$$\frac{(\alpha-1)^{2\rho_h(g)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(g)}\cdot\alpha^{\rho_h(g)+1}} \cdot \frac{\sigma_h(fog)}{\sigma_h(g)} \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)}.$$

Using the notion of relative weak type, we may state the following two theorems without proof because those may be carried out with the help of Lemma 3 and Lemma 5 and in the line of Theorem 15 and Theorm 16 respectively.

Theorem 17 Let f, g and h be any three entire functions such that $(i)\ 0<\lambda_h(f)<\infty$, $(ii)\ 0<\tau_h(f)<\infty$, $(iii)\ \lambda_h(fog)=\lambda_h(f)$ and $(iv)\ \tau_h(fog)<\infty$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\le\frac{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(f)}\cdot\alpha^{\lambda_h(f)+1}}{(\alpha-1)^{2\lambda_h(f)}}\cdot\frac{\tau_h(fog)}{\tau_h(f)}$$
and
$$\frac{(\alpha-1)^{2\lambda_h(f)}}{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(f)}\cdot\alpha^{\lambda_h(f)+1}}\cdot\frac{\tau_h(fog)}{\tau_h(f)}\le\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}.$$

Theorem 18 Let f, g and h be any three entire functions such that $(i)\ 0<\lambda_h(g)<\infty$, $(ii)\ 0<\tau_h(g)<\infty$, $(iii)\ \lambda_h(fog)=\lambda_h(g)$ and $(iv)\ \tau_h(fog)<\infty$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)}\le\frac{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(g)}\cdot\alpha^{\lambda_h(g)+1}}{(\alpha-1)^{2\lambda_h(g)}}\cdot\frac{\tau_h(fog)}{\tau_h(g)}$$
and
$$\frac{(\alpha-1)^{2\lambda_h(g)}}{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(g)}\cdot\alpha^{\lambda_h(g)+1}}\cdot\frac{\tau_h(fog)}{\tau_h(g)}\le\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)}.$$

Similarly one may state the following four theorems without proof on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum modulus :

Theorem 19 Let f, g and h be any three entire functions with $(i)\ 0<\rho_h^{L^*}(f)<\infty$, $(ii)\ 0<\sigma_h^{L^*}(f)<\infty$, $(iii)\ \rho_h^{L^*}(fog)=\rho_h^{L^*}(f)$ and $(iv)\ \sigma_h^{L^*}(fog)<\infty$. Then

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)}\le\frac{\sigma_h^{L^*}(fog)}{\sigma_h^{L^*}(f)}\le\limsup_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)}.$$

Theorem 20 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h^{L^*}(g)<\infty$, $(ii)\ 0<\sigma_h^{L^*}(g)<\infty$, $(iii)\ \rho_h^{L^*}(fog)=\rho_h^{L^*}(g)$ and $(iv)\ \sigma_h^{L^*}(fog)<\infty$. Then

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)}\le\frac{\sigma_h^{L^*}(fog)}{\sigma_h^{L^*}(g)}\le\limsup_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)}.$$

Theorem 21 Let f, g and h be any three entire functions with $(i)\ 0<\lambda_h^{L^*}(f)<\infty$, $(ii)\ 0<\tau_h^{L^*}(f)<\infty$, $(iii)\ \lambda_h^{L^*}(fog)=\lambda_h^{L^*}(f)$ and $(iv)\ \tau_h^{L^*}(fog)<\infty$. Then

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)}\le\frac{\tau_h^{L^*}(fog)}{\tau_h^{L^*}(f)}\le\limsup_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)}.$$

Theorem 22 Let f, g and h be any three entire functions such that $(i)\ 0<\lambda_h^{L^*}(g)<\infty$, $(ii)\ 0<\tau_h^{L^*}(g)<\infty$, $(iii)\ \lambda_h^{L^*}(fog)=\lambda_h^{L^*}(g)$ and $(iv)\ \tau_h^{L^*}(fog)<\infty$. Then

$$\liminf_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)}\le\frac{\tau_h^{L^*}(fog)}{\tau_h^{L^*}(g)}\le\limsup_{r\to\infty}\frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)}.$$

Theorem 23 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h(f)<\infty$, $(ii)\ \sigma_h(f)<\infty$, $(iii)\ \lambda_h(fog)=\rho_h(f)$ and $(iv)\ \tau_h(fog)>0$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\ge\frac{(\alpha-1)^{2\rho_h(f)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}}\cdot\frac{\tau_h(fog)}{\sigma_h(f)}.$$

Proof. From the definition of $\tau_h(fog)$ and in view of Lemma 4 and Lemma 5 we obtain for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{fog}(r)\ge M_h^{-1}M_{fog}\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}r\right)$$

$$i.e.,\ \mu_h^{-1}\mu_{fog}(r)\ge(\tau_h(fog)-\varepsilon)\left\{\left(\frac{(\alpha-1)r}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)\right\}^{\lambda_h(fog)}$$

$$i.e.,\ \mu_h^{-1}\mu_{fog}(r)\ge\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\lambda_h(fog)}\cdot(\tau_h(fog)-\varepsilon)\cdot r^{\lambda_h(fog)}.\tag{15}$$

Thus from (8) and (15) we get for all sufficiently large values of r that

$$\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\ge\frac{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\lambda_h(fog)}\cdot(\tau_h(fog)-\varepsilon)\cdot r^{\lambda_h(fog)}}{\alpha\,(\sigma_h(f)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(f)}}.\tag{16}$$

Since $\lambda_h(fog)=\rho_h(f)$, we obtain from (16) that

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\ge\frac{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\rho_h(f)}}{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)^{\rho_h(f)}}\cdot\frac{(\tau_h(fog)-\varepsilon)}{\alpha\,(\sigma_h(f)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\ge\frac{(\alpha-1)^{2\rho_h(f)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(f)}\cdot\alpha^{\rho_h(f)+1}}\cdot\frac{\tau_h(fog)}{\sigma_h(f)}.$$

Thus the theorem is established.

Theorem 24 Let f, g and h be any three entire functions such that $(i)\ 0<\lambda_h(f)<\infty$, $(ii)\ \tau_h(f)<\infty$, $(iii)\ \rho_h(fog)=\lambda_h(f)$ and $(iv)\ \sigma_h(fog)>0$. Then for any $\alpha>1$ and $\gamma>0$,

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\le\frac{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(f)}\cdot\alpha^{\lambda_h(f)+1}}{(\alpha-1)^{2\lambda_h(f)}}\cdot\frac{\sigma_h(fog)}{\tau_h(f)}.$$

Proof. From the definition of $\tau_h(f)$ and in view of Lemma 4 and Lemma 5 we obtain for all sufficiently large values of r that

$$\mu_h^{-1}\mu_{fog}(r)\ge M_h^{-1}M_f\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}r\right)$$

$$i.e.,\ \mu_h^{-1}\mu_f(r)\ge(\tau_h(f)-\varepsilon)\left\{\left(\frac{(\alpha-1)r}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)\right\}^{\lambda_h(f)}$$

$$i.e.,\ \mu_h^{-1}\mu_f(r)\ge\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\lambda_h(f)}\cdot(\tau_h(f)-\varepsilon)\cdot r^{\lambda_h(f)}.\tag{17}$$

Thus from (7) and (17) we get for all sufficiently large values of r that

$$\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\le\frac{\alpha\,(\sigma_h(fog)+\varepsilon)\left\{\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)r\right\}^{\rho_h(fog)}}{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\lambda_h(f)}\cdot(\tau_h(f)-\varepsilon)\cdot r^{\lambda_h(f)}}.\tag{18}$$

In view of the condition (iii) we get from (18) that

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\le\frac{\alpha\,(\sigma_h(fog)+\varepsilon)\left(\frac{\alpha+\gamma\alpha-\gamma}{\alpha-1}\right)^{\lambda_h(f)}}{\left(\frac{(\alpha-1)}{(\alpha+\gamma\alpha-\gamma)\alpha}\right)^{\lambda_h(f)}\cdot(\tau_h(f)-\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_f(r)}\le\frac{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(f)}\cdot\alpha^{\lambda_h(f)+1}}{(\alpha-1)^{2\lambda_h(f)}}\cdot\frac{\sigma_h(fog)}{\tau_h(f)}.$$

Thus the theorem follows from above.

In the line of Theorem 23 and Theorem 24 we may state the following two theorems without proof :

Theorem 25 Let f, g and h be any three entire functions such that $(i)\ 0<\rho_h(g)<\infty$, $(ii)\ \sigma_h(g)<\infty$, $(iii)\ \lambda_h(fog)=\rho_h(g)$ and $(iv)\ \tau_h(fog)>0$. Then for any $\alpha>1$ and $\gamma>0$,

$$\liminf_{r\to\infty}\frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)}\ge\frac{(\alpha-1)^{2\rho_h(g)}}{(\alpha+\gamma\alpha-\gamma)^{2\rho_h(g)}\cdot\alpha^{\rho_h(g)+1}}\cdot\frac{\tau_h(fog)}{\sigma_h(g)}.$$

Theorem 26 Let f , g and h be any three entire functions such that $(i)\ 0 < \lambda_h(g) < \infty$, $(ii)\ \tau_h(g) < \infty$, $(iii)\ \rho_h(fog) = \lambda_h(g)$ and $(iv)\ \sigma_h(fog) > 0$. Then for any $\alpha > 1$ and $\gamma > 0$,

$$\limsup_{r \to \infty} \frac{\mu_h^{-1}\mu_{fog}(r)}{\mu_h^{-1}\mu_g(r)} \leq \frac{(\alpha+\gamma\alpha-\gamma)^{2\lambda_h(g)} \cdot \alpha^{\lambda_h(g)+1}}{(\alpha-1)^{2\lambda_h(g)}} \cdot \frac{\sigma_h(fog)}{\tau_h(g)} .$$

Analogously we may also state the following four theorems without proof on the basis of relative type and relative weak type of entire functions with respect to another entire functions in terms of their maximum modulus :

Theorem 27 Let f , g and h be any three entire functions such that $(i)\ 0 < \rho_h(f) < \infty$, $(ii)\ \sigma_h(f) < \infty$, $(iii)\ \lambda_h(fog) = \rho_h(f)$, and $(iv)\ \tau_h(fog) > 0$. Then

$$\liminf_{r \to \infty} \frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \geq \frac{\tau_h(fog)}{\sigma_h(f)} .$$

Theorem 28 Let f , g and h be any three entire functions such that $(i)\ 0 < \lambda_h(f) < \infty$, $(ii)\ \tau_h(f) < \infty$, $(iii)\ \rho_h(fog) = \lambda_h(f)$, and $(iv)\ \sigma_h(fog) > 0$. Then

$$\limsup_{r \to \infty} \frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_f(r)} \leq \frac{\sigma_h(fog)}{\tau_h(f)} .$$

Theorem 29 Let f, g and h be any three entire functions such that $(i)\ 0 < \rho_h(g) < \infty$, $(ii)\ \sigma_h(g) < \infty$, $(iii)\ \lambda_h(fog) = \rho_h(g)$, and $(iv)\ \tau_h(fog) > 0$. Then

$$\liminf_{r \to \infty} \frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)} \geq \frac{\tau_h(fog)}{\sigma_h(g)} .$$

Theorem 30 Let f, g and h be any three entire functions such that $(i)\ 0 < \lambda_h(g) < \infty$, $(ii)\ \tau_h(g) < \infty$, $(iii)\ \rho_h(fog) = \lambda_h(g)$, and $(iv)\ \sigma_h(fog) > 0$. Then

$$\limsup_{r \to \infty} \frac{M_h^{-1}M_{fog}(r)}{M_h^{-1}M_g(r)} \leq \frac{\sigma_h(fog)}{\tau_h(g)} .$$

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