On Convergence Theorem of fixed point theorem for Nonself I-Nonexpansive Mapping of Ishikawa Iteration in Banach Spaces

Poonam L. Sagar[1*], S. K. Malhotra[2#]

[1]Assistant Professor, Department of Applied Mathematics and Computer Science, SATI (Deg), Vidisha, Madhya Pradesh, INDIA.

[2]HOD, Department of Mathematics, Govt. S. G. S. P.G. College Ganj Basoda, Distt Vidisha, Madhya Pradesh, INDIA.

Corresponding Addresses:

[#]skmalhotra75@gmail.com , [*]poonamlata.sagar@gmail.com

Research Article

Abstract: Suppose that E be a uniformly convex Banach Space, Let K be a nonempty convex subset of E with P as a nonexpansive retraction. Let T:K→ E be a given nonself mapping. The modified Ishikawa iterative scheme {xn} is defined by (1.8). We establish the weak convergence of a sequence of a modified ishikawa iteration of a nonself I- nonexpansive mapping in a banach space which satisfies Opial's condition.

Keywords: Mann iteration, Ishikawa iteration and Noor iteration, nonself nonexpansive mapping.

Mathematics Subject Classification: 47H09, 47H10.

1. Introduction and preliminaries

Let E be a Banach space, K a nonempty, convex subset of E, and T a self map of K. Three most popular iteration procedures for obtaining fixed points of T , if they exist, are Mann iteration [12], defined by

$$u1 \in K, un+1 = (1 - \alpha n)un + \alpha\, nTun, \quad n \geq 1 , \qquad (1.1)$$

and Ishikawa iteration [8], defined by

$$x1 \in K, xn+1 = (1 - \alpha n)xn + \alpha nTyn$$
$$yn = (1 - \beta n)xn + \beta nTxn , \quad n \geq 1 , \qquad (1.2)$$

and Noor iteration [9], dfined by

$$xn+1 = (1 - \alpha n)xn + \alpha nTyn$$
$$yn = (1 - \beta n)xn + \beta nTzn$$
$$zn = (1- \gamma n) xn + \gamma n Txn , \quad n\geq 1, \qquad (1.3)$$

for certain choices of {αn} , {βn}, {γn}⊂ [0, 1] .

In the above taking βn = 0 in (1.2) we obtain iteration (1.1).

Let K be a closed convex bounded subset of uniformly convex Banach space E = (E, $\|\cdot\|$) and T self-mappings of E . Then T is called nonexpansive on K if

$$\|Tx - Ty\| \leq \|x - y\| \qquad (1.4)$$

for all x, y ∈ K . Let F(T) : = { x ∈ K : Tx = x} be denoted as the set of fixed points of a mapping T.

The first nonlinear ergodic theorem was proved by Baillon [4] for general nonexpansive mappings in Hilbert space H : if K is a closed and convex subset of H and T has a fixed point, then for every x ∈ K , {Tn x} is weakly almost convergent, as n →∞ , to a fixed point of T . It was also shown by Pazy [1] that if H is a real Hilbert space and (1/n)$\sum_{i=0}^{n-1} T$ ix converges weakly, as n →∞ ,to y ∈ K , then y ∈ F(T). The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [5] and Dotson [11] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [10] in metric spaces which we adapt to a normed space as follows: T is called a quasi-nonexpansive mapping provide

$$\|Tx - f\| \leq \|x - f\| \qquad (1.5)$$

for all x ∈ K and f ∈ F(T).

Recall that a Banach space E is said to satisfy Opial's condition [14] if, for each sequence {xn} in E , the condition xn → x implies that

$$\overline{\lim_{n\to\infty}}\|x_n - x\| < \overline{\lim_{n\to\infty}}\|x_n - y\| \qquad (1.6)$$

for all y ∈ E with y ≠ x . It is well known from [14] that all lp spaces for 1 <p< ∞ have this property. However, the Lp spaces do not, unless p =2. The following definitions and statements will be needed for the proof of our theorem. Let K be a subset of a normed space E = (E, $\|\cdot\|$) and T and I self- mappings of K . Then T is called I -nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \qquad (1.7)$$

for all x, y ∈ K [7].

T is called I -quasi-nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \qquad (1.8)$$

for all x, y ∈ K and f ∈ F(T) ∩ F(I).

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map P: E → K such that Px = x for all x ∈ K. A map P : E → E is said to be a retraction if P2= P . It follows that if a map P is a retraction, then Py = y for all y in the range of P. A set K is optimal if each point outside K can be moved to be closer to all points of K. Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

Remark 1.1.From the above definitions it is easy to see that if F(T) is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive. There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn andWilliamson [13]. Their analysis was related to the convergence of Mann iterates studied by Dotson [11].Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [6]. In [9], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [3], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized. In [2], Rhoades and Temir considered T and I self-mappings of K , where T is an I-nonexpansive mapping. They established the weak convergence of the sequence of Mann iterates to a common fixed point of T and I . More precisely, they proved the following theorems.

Theorem (Rhoades and Temir [2]): Let K be a closed convex bounded subset of uniformly convex Banach space E , which satisfies Opial's condition, and let T , I self-mappings of K with T be an I -nonexpansive mapping, I a nonexpansive on K . Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Ishikawa iterates converges weakly to common fixed point of F(T) ∩ F(I). In the above theorem, T remains self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain K of T is a proper subset of E and T maps K into E then, the iteration formula (1.1) may fail to be well defined. One method that has been used to overcome this in the case of single operator T is to introduce a retraction P : E → K in the recursion formula (1.1) as follows:

$$u_1 \in K, \quad u_{n+1} = (1-\alpha_n)u_n + \alpha_n PTu_n, \quad n \geq 1.$$

In this paper, we consider T and I nonself mappings of K , where T is an I-nonexpansive mappings. We establish the weak convergence of the sequence of modified Ishikawa iterates to a common fixed point of T and I .

Let E be a uniformly convex Banach space, let K be a nonempty convex subset of E with P as a nonexpansive retraction. Let T : K → E be a given nonself mapping. The modified Ishikawa iterative scheme $\{x_n\}$ is defined by

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n Ty_n)$$
$$y_n = P((1-\beta_n)x_n + \beta_n Tx_n), \quad n \geq 1,$$

Let E be a uniformly convex Banach space, let K be a nonempty convex subset of E with P as a nonexpansive retraction. Let T: K → E be a given nonself mapping. The Noor iterative scheme

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n$$
$$y_n = (1-\beta_n)x_n + \beta_n Tz_n$$
$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \quad n \geq 1, \qquad (1.8)$$

for certain choices of $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset (0, 1)$. Clearly, if T is self maps, then (1.8) reduces to an iteration scheme (1.2).

2. The main result

Theorem 2.1. Let K be a closed convexbounded subset of uniformly convex Banach space E, which satisfies Opial's condition, and let T , I non- selfmappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of modified Nooriterates converges weakly to common fixed point of F(T) ∩ F(I).

Proof. If F(T) ∩ F(I) is nonempty and a singleton, then the proof is complete . We will assume that F(T) ∩ F(I) is nonempty and that F(T) ∩ F(I) is not a singleton.

$$\|x_{n+1} - f\| = \|P((1-\alpha_n)x_n + \alpha_n Ty_n) - f\|$$
$$= \|P(1-\alpha_n)x_n + \alpha_n TPy_n - Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|TPy_n - Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|TP(1-\beta_n)x_n + \beta_n Tz_n - Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n\|(1-\beta_n)x_n + \beta_n Tz_n - f\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n)\|x_n - f\| + \alpha_n\beta_n\|Tz_n - Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n)\|x_n - f\| + \alpha_n\beta_n\|TP(1-\gamma_n)x_n + \gamma_n Tx_n - Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n)\|x_n - f\| + \alpha_n\beta_n\|(1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n+\gamma_n)Tf\|$$
$$\leq (1-\alpha_n)\|x_n - f\| + \alpha_n(1-\beta_n)\|x_n - f\| + \alpha_n\beta_n(1-\gamma_n)\|x_n - f\| + \alpha_n\beta_n\gamma_n\|x_n - f\|$$
$$= \|x_n - f\| \qquad (2.1)$$

Thus for $\alpha_n \neq 0$, $\beta_n \neq 0$ and $\gamma_n \neq 0$, $\{\|x_n - f\|\}$ is a nonincreasing sequence.

Then $\lim_{n\to\infty}\|x_n - f\|$ exits.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I . The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K .Let $\{x_{nk}\}$ and $\{x_{mk}\}$ be two subsequences of $\{x_n\}$ which converge weakly to f and q , respectively. We will show that f = q Suppose that E satisfies Opial's condition and that f ≠ q is in weak limit set of the sequence $\{x_n\}$ .Then $\{x_{nk}\} \to f$ and $\{x_{mk}\} \to q$, respectively. Since $\lim_{n\to\infty}\|x_n - f\|$ exists for any f ∈ F(T) ∩ F(I) by Opial's condition, we conclude that

$$\lim_{n\to\infty}\|x_n - f\| = \lim_{k\to\infty}\|x_{nk} - f\| < \lim_{k\to\infty}\|x_{nk} - q\| < \lim_{j\to\infty}\|x_{mj} - f\| = \lim_{n\to\infty}\|x_n - f\|.$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of F(T) ∩ F(I).

References

1. Pazy, On the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space, Israel Journal of Mathematics 26 (1977), no. 2,197-204.
2. B. E. Rhoades and S. Temir, Convergence theorems for I-nonexpansive mapping, to appear in International Journal of Mathematics and Mathematical Sciences.
3. H. Zhou, R. P. Agarwal, Y. J. Cho, and Y. S. Kim, Nonexpansive mappings and iterative methods in uniformly convex Banach spaces, Georgian Mathematical Journal 9 (2002), no. 3, 591-600.
4. J. B. Baillon, Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert, Comptes Rendus de l'Academie des Sciences de Paris, Serie A 280(1975), no. 22, 1511-1514.
5. J. B. Diaz and F. T. Metcalf, On the set of subsequential limit points of successive approximations, Transactions of the American Mathematical Society 135(1969), 459-485.
6. M. K. Ghosh and L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, Journal of Mathematical Analysis and Applications 207(1997), no. 1,96-103.
7. N. Shahzad, Generalized I-nonexpansive maps and best approximations in Banach spaces, Demon-stratio Mathematica 37 (2004), no. 3, 597-600.
8. S. Ishikawa, Fixed points by a new iteration method, Proc. Am. Math. Soc. 44 (1974) 147-150.
9. S. Temir and O. Gul, Convergence theorem for I-asymptotically quasi-nonexpansive mapping in Hilbert space, Journal of Mathematical Analysis and Applications 329 (2007) 759-765.
10. W. A. Kirk, Remarks on approximation and approximate fixed points in metric fixed point theory, Annales Universitatis Mariae Curie-Sklodowska. Section A 51(1997), no. 2,167-178.
11. W. G. Dotson Jr., On the Mann iterative process, Transactions of the American Mathematical Society 149(1970), no. 1,65-73.
12. W. R. Mann, Mean value methods in iteration, Proc. Am. Math. Soc. 4 (1953) 506-510.
13. W. V. Petryshyn and T. E. Williamson Jr., Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings, Journal of Mathematical Analysis and Applications 43 (1973), 459-497.
14. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bulletin of the American Mathematical Society 73 (1967), 591-597.