

# Extension of Some Common Fixed Point Theorems for Non Compatible Mappings in Fuzzy Metric Spaces

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## Research Article

**Abstract:** This paper represents the concept of non compatibility and this is extension of some results for non compatible mappings in fuzzy metric space. We have followed the condition of continuity and t-norm.

**Key words:** Fuzzy metric space, Common fixed point, Non compatible mapping.

### 1 Introduction and Preliminaries

The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [18] in 1965, as a new way to represent the vagueness in every day life. In mathematical programming, problems are expressed as optimizing some goal function given certain constraints, and there are real life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution, that is quite useful, is the one using fuzzy sets [17]. The concept of fuzzy metric space has been introduced and generalized by many ways ([4], [7]). George and Veeramani ([5]) modified the concept of fuzzy metric space introduced by Kramosil and Michalek [8]. They also obtained a Hausdorff topology for this kind of fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and  $\_1$  theory (see, [12] and references mentioned therein). Many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces ([10], [13], [16]). Regan and Abbas [14] obtained some necessary and sufficient conditions for the existence of common fixed point in fuzzy metric spaces. Recently, Cho et al [3] established some fixed point theorems for mappings satisfying generalized contractive condition in fuzzy metric space. The aim of this paper is to obtain common fixed point of mappings satisfying generalized contractive type conditions without exploiting the notion of continuity in the setting of fuzzy metric spaces. Our results generalize several comparable results in existing literature [2].

**Definition 1.1** ([18]) Let  $X$  be any set. A fuzzy set  $A$  in  $X$  is a function with domain  $X$  and values in  $[0, 1]$ .

**Definition 1.2** ([15]) A mapping  $*$ :  $[0, 1] \times [0, 1] \rightarrow [0, 1]$  is called a continuous t-norm if  $([0, 1], *)$  is an abelian topological monoid with unit 1 such that  $a * b \leq c * d$ , for  $a \leq c, b \leq d$ . Examples of t-norms are  $a * b = \min\{a, b\}$  (minimum t-norm),  $a * b = ab$  (product t-norm), and  $a * b = \max\{a+b-1, 0\}$  (Lukasiewicz t-norm).

**Definition 1.3** ([8]) The 3-tuple  $(X, M, *)$  is called a fuzzy metric space if  $X$  is an arbitrary set,  $*$  is a continuous t-norm and  $M$  is a fuzzy set in  $X^2 \times [0, \infty)$  satisfying the following conditions:

- $M(x, y, t) > 0$ ,
- $M(x, y, t) = 1$  for all  $t > 0$  if and only if  $x = y$
- $M(x, y, t) = M(y, x, t)$ ,
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ ,
- $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$  is a continuous function, for all  $x, y, z \in X$  and  $t, s > 0$ .

Note that,  $M(x, y, t)$  can be thought of as the definition of nearness between  $x$  and  $y$  with respect to  $t$ . It is known that  $M(x, y, \cdot)$  is nondecreasing for all  $x, y \in X$  [5].

Let  $(X, M, *)$  be a fuzzy metric space. For  $t > 0$ , the open ball  $B(x, r, t)$  with center  $x \in X$  and radius  $0 < r < 1$  is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

The collection  $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$  is a neighborhood system for a topology  $T$  on  $X$  induced by the fuzzy metric  $M$ . This topology is Hausdorff and first countable.

A sequence  $\{x_n\}$  in  $X$  converges to  $x$  ([6]) if and only if for each  $\epsilon > 0$  and

each  $t > 0$  there exists  $n_0 \in \mathbb{N}$

$$M(x_n, x, t) > 1 - \epsilon$$

for all  $n \geq n_0$ .

**Lemma 1.4** ([11]) If, for all  $x, y \in X, t > 0$ , and for a number  $q \in (0, 1), M(x, y, qt) \geq M(x, y, t)$ , then  $x = y$ .

**Lemma 1.5** ([5]) Let  $(X, M, *)$  be a fuzzy metric space. Then  $M$  is a continuous function on  $X^2 \times (0, \infty)$ .

**Definition 1.6** ([16]) Let  $f$  and  $g$  be self maps on a fuzzy metric space  $(X, M, *)$ .

They are compatible or asymptotically commuting if for all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} M(fg_{x_n}, g_{x_n}, t) = 1$$

whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = z$ , for some  $z \in X$ . Mappings  $f$  and  $g$  are noncompatible maps, if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} f_{x_n} = p = \lim_{n \rightarrow \infty} g_{x_n}$ , but either  $\lim_{n \rightarrow \infty} M(fg_{x_n}, g_{x_n}, t) \neq 1$  or the limit does not exist for all  $p \in X$ .

**Definition 1.7** ([3]) Let  $f$  and  $g$  be self maps on a fuzzy metric space  $(X, M, *)$ .

A pair  $\{f, g\}$  is said to be:

(f) compatible of type (I) if for all  $t > 0$ ,

$$\lim_{n \rightarrow \infty} M(fg_{x_n}, x, t) \leq M(gx, x, t)$$

whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = x$ , for some  $x \in X$ .

(g) compatible of type (II) if the pair  $(g, f)$  is compatible of type (I).

**Definition 1.8** Mappings  $f$  and  $g$  from a fuzzy metric space  $(X, M, *)$  into itself are weakly compatible if they commute at their coincidence point, that is  $fx = gx$  implies that  $fgx = gfx$ .

It is known that a pair  $\{f, g\}$  of compatible maps is weakly compatible but converse is not true in general.

**Definition 1.9** Let  $f$  and  $g$  be self maps on a fuzzy metric space  $(X, M, *)$ . They are said to satisfy (EA) property if there exists a sequence  $\{x_n\}$  in  $X$  such that

$$\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = x \text{ for some } x \in X.$$

**Definition 1.10** Mappings  $A, B, S$  and  $T$  on a fuzzy metric space  $(X, M, *)$  are said to satisfy common (EA) property if there exists sequences  $\{x_n\}$  and  $\{y_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = x$  for some  $x \in X$ .

For more on (EA) and common (EA) properties, we refer to [1] and [9]. Note that compatible, noncompatible, compatible of type (I) and compatible of type (II) satisfy (EA) property but converse is not true in general.

**Example 1.11** Let  $(X, M, *)$  be a fuzzy metric space, where  $X = [0, 2]$  with minimum  $t$ -norm, and  $M(x, y, t) = t / (t + d(x, y))$  for all  $t > 0$  and for all  $x, y \in X$ . Define the self maps  $f$  and  $g$  as follows:

$$fx = \begin{cases} 2, & \text{when } x \in [0, 1] \\ x/2 & \text{when } 1 < x \leq 2 \end{cases}$$

$$gx = \begin{cases} 0, & \text{when } x = 1 \\ (x+3)/5, & \text{otherwise.} \end{cases}$$

Now, suppose  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = z$ . By definition of  $f$  and  $g$ , we have  $z \in \{1\}$ . Thus  $\{f, g\}$  satisfies (EA) property. Note that  $\{f, g\}$  is not compatible. Indeed, if  $\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = 1$ ,

then it must be  $x_n \rightarrow 2^-$  and so  $\lim_{n \rightarrow \infty} gf_{x_n} = 4/5$  and  $\lim_{n \rightarrow \infty} fg_{x_n} = 2$ . Therefore  $\lim_{n \rightarrow \infty} M(fg_{x_n}, g_{x_n}, t) = M(2, 4/5, t) = t / (t + (6/5)) < 1$ ,

for all  $t > 0$ . Also note that,  $\{f, g\}$  is not compatible of type (II). Since

$$\lim_{n \rightarrow \infty} M(fg_{x_n}, x, t) = M(4/5, 1, t)$$

$$= t / (t + (1/5)) > M(fx, x, t) = M(2, 1,$$

$$t) = t / (1+t)$$

for all  $t > 0$ .

**Example 1.12** Let  $(X, M, *)$  be a fuzzy metric space, where  $X = [0, 1]$  with minimum  $t$ -norm, and  $M(x, y, t) = t / (t + d(x, y))$  for all  $t > 0$  and for all  $x, y \in X$

Define the self map  $g$  as follows:

$$gx = \begin{cases} 1/2, & \text{when } 0 \leq x < 1/2 \text{ or } x = 1 \\ 1, & \text{when } 1/2 \leq x < 1. \end{cases}$$

Let  $f$  be the identity map. Then, as  $\{f, g\}$  is commuting, it is compatible and hence satisfy property (EA). However,  $\{f, g\}$  is not compatible of type (I). Indeed, suppose  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} f_{x_n} = \lim_{n \rightarrow \infty} g_{x_n} = z$ . By definition of  $f$  and  $g$ , we have  $z \in \{1/2, 1\}$ .

Now if  $z = 1/2$ , we can consider  $x_n = 1/2 - (1/n)$

Therefore,  $\lim_{n \rightarrow \infty} M(fg_{x_n}, z, t) = M(1/2, 1/2, 1) = 1 > t / (1/2 + t) = M(gz, z, t)$ , for all  $t > 0$ .

if  $z = 1$ , we can consider  $x_n = 1 - 1/n$ . Therefore,  $\lim_{n \rightarrow \infty} M(fg_{x_n}, z, t) = M(1, 1, t) = 1 > t / (1/2 + t) = M(gz, z, t)$ , for all  $t > 0$ .

Let a class of implicit relations be the set of all continuous functions  $\phi : [0, 1] \times [0, 1] \rightarrow [0, 1]$  which are increasing in each coordinate and  $\phi(t, t, t, t) > t$  for all  $t \in [0, 1]$

## 2 Main Results:

The following result provides necessary conditions for the existence of common fixed point of noncompatible maps in a Fuzzy metric space.

**Theorem 2.1** Let  $(X, M, *)$  be a fuzzy metric space. Let  $A, B, S$  and  $T$  be maps from  $X$  into itself with  $A(X) \subseteq T(X)$  and  $B(X) \subseteq S(X)$  and there exists a constant  $k \in (0, 1/2)$  such that

$$M(Ax, By, kt) \geq \phi(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, \alpha t), M(By, Sx, (2 - \alpha)t), M(Tx, By, \alpha t)) \quad (1)$$

for all  $x, y \in X$ ,  $\alpha \in (0, 2)$ ,  $t > 0$  and  $\phi \in \Phi$ . Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$  provided the pair  $\{A, S\}$  or  $\{B, T\}$  satisfies (EA) property, one of  $A(X), T(X), B(X), S(X)$  is a closed subset of  $X$  and the pairs  $\{B, T\}$  and  $\{A, S\}$  are weakly compatible.

**Proof.** Suppose that a pair  $\{B, T\}$  satisfies property (EA), therefore there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} Bx_n = z = \lim_{n \rightarrow \infty} Tx_n$ .

Now  $B(X) \subseteq S(X)$  implies that there exists a sequence  $\{y_n\}$  in  $X$  such that  $Bx_n = Sy_n$ . For  $\alpha = 1, x = y_n$  and  $y = x_n, (1)$  becomes

$$M(Ay_n, Bx_n, kt) \geq \phi \left( M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Ay_n, Tx_n, t), M(Bx_n, Sy_n, t), M(Ty_n, Bx_n, \alpha t) \right)$$

Taking limit  $n \rightarrow \infty$ , we obtain

$$M(\lim_{n \rightarrow \infty} Ay_n, z, kt) \geq \phi \left( M(z, z, t), M(\lim_{n \rightarrow \infty} Ay_n, z, t), M(\lim_{n \rightarrow \infty} Ay_n, z, t), M(z, z, t), M(z, z, t) \right)$$

Since  $\phi$  is increasing in each of its coordinate and  $\_ (t, t, t, t) > t$  for all  $t \in [0, 1)$ ,  $M(\lim_{n \rightarrow \infty} Ay_n, z, kt) > M(\lim_{n \rightarrow \infty} Ay_n, z, t)$  which by Lemma 1.4 implies that

$\lim_{n \rightarrow \infty} Ay_n = z$ . Suppose that  $S(X)$  is a closed subspace of  $X$ . Then,  $z = Su$  for some  $u \in X$ . Now replacing  $x$  by  $u$  and  $y$  by  $x_{2n+1}$ , and  $\alpha = 1$  in (1) we have

$$M(Au, Bx_{2n+1}, kt) \geq \phi \left( M(Su, Tx_{2n+1}, t), M(Au, Su, t), M(Bx_{2n+1}, Tx_{2n+1}, t), M(Au, Tx_{2n+1}, t), M(Bx_{2n+1}, Su, t), M(Tu, Bx_{2n+1}, \alpha t) \right)$$

$$M(Au, z, kt) \geq \phi \left( M(z, z, t), M(Au, z, t), M(z, z, t), M(Au, z, t), M(z, z, t), M(z, z, t) \right) > M(Au, z, t)$$

which implies that  $Au = z$ . Hence  $Au = z = Su$ . Since,  $A(X) \subseteq T(X)$ , there exist  $v \in X$  such that  $z = Tv$ . Following the arguments similar to those given above we obtain  $z = Bv = Tv$ . Since  $u$  is coincidence point of the pair  $\{A, S\}$ , therefore  $SAu = ASu$ , and  $Az = Sz$ . Now we claim that  $Az = z$ , if not, then using (1) with  $\alpha = 1$ , we arrive at

$$M(Az, z, kt) = M(Az, Bv, kt) \geq \phi \left( M(Sz, Tv, t), M(Az, Sz, t), M(Bv, Tv, t), M(Az, Tv, t), M(Bv, Sz, t), (Az, Tv, t) \right) > M(Az, z, t),$$

a contradiction. Hence  $z = Az = Sz$ . Similarly, we can prove that  $z = Bz = Tz$ . The uniqueness of  $z$  follows from (1). Following Theorem was proved in [3]: Let

$(X, M, *)$  be a fuzzy metric space with  $t * t = t$ . Let  $A, B, S$  and  $T$  be maps from  $X$  into itself with  $A(X) \subseteq T(X)$  and  $B(X) \subseteq S(X)$  and there exists a constant  $k \in (0, 1/2)$ ,  $\alpha \in (0, 2)$  such that

$$M(Ax, By, kt) \geq \phi \left( M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \right.$$

$$\left. M(Ax, Ty, \alpha t), M(By, Sx, (2 - \alpha)t), M(Tx, By, \alpha t) \right) \quad (2)$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi$ . Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$  provided the pair  $\{A, S\}$  and  $\{B, T\}$  are compatible of type (II), and  $A$  or  $B$  are continuous or the pair  $\{A, S\}$  and  $\{B, T\}$  are compatible of type (I), and  $S$  or  $T$  are continuous.

**Theorem 2.3** Let  $(X, M, *)$  be a fuzzy metric space. Let  $A, B, S$  and  $T$  be maps from  $X$  into itself such that  $M(Ax, By, kt) \geq \phi \left( M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Sx, \alpha t), M(By, Sx, (2 - \alpha)t) \right)$  (3)

for all  $x, y \in X, k \in (0, 1/2), \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi$ . Then  $A, B, S$  and  $T$  have a unique common fixed point in  $X$  provided the pair  $\{A, S\}$  and  $\{B, T\}$  satisfy common (EA) property,  $T(X)$ , and  $S(X)$  are closed subset of  $X$  and the pairs  $\{B, T\}$  and  $\{A, S\}$  are weakly compatible.

**Proof.** Suppose that  $(A, S)$  and  $(B, T)$  satisfy a common (EA) property, there exist two sequences  $\{x_n\}$  and  $\{y_n\}$  such that  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = z$  for some  $z$  in  $X$ . Since  $S(X)$  and  $T(X)$  are closed subspace of  $X$ , therefore  $z = Su = Tv$  for some  $u, v \in X$ . Now we claim that  $Au = z$ . For this, replace  $x$  by  $u$  and  $y$  by  $y_n$  in (3) with  $\alpha = 1$ , we obtain

$$M(Au, By_n, kt) \geq \phi \left( M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, t), M(Au, Ty_n, t), M(By_n, Su, t) \right)$$

which on taking  $n \rightarrow \infty$  gives

$$M(Au, z, kt) > M(Au, z, t)$$

Hence  $Au = z = Su$ . Again using (3) with  $\alpha = 1$ ,  $M(Tv, Bv, kt) = M(Au, Bv, kt)$

$$\geq \phi \left( M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Au, Tv, t), M(Bv, Su, t) \right) > M(Tv, Bv, t),$$

which implies that  $Tv = Bv$  and hence  $Au = z = Su = Bv = Tv$ . The rest of the proof follows as in Theorem 2.1.

**Corollary 2.4** Let  $(X, M, *)$  be a fuzzy metric space, where  $*$  is any continuous  $t$ - norm. Let  $A, B, R, S, H$  and  $T$  be mappings from  $X$  into itself with  $A(X) \subseteq TH(X), B(X) \subseteq SR(X)$  and there exists a constant  $k \in (0, 1/2)$  such that

$$M(Ax, By, kt) \geq \phi \left( M(SRx, THy, t), M(Ax, SRx, t), M(By, THy, t), M(Ax, THy, \alpha t), M(By, SRx, (2 - \alpha)t) \right)$$

for all  $x, y \in X, \alpha \in (0, 2), t > 0$  and  $\phi \in \Phi$ . Then  $A, B, R, S, H$  and  $T$  have a unique common fixed point in  $X$  provided the pair  $\{A, SR\}$  or  $\{B, TH\}$  satisfies (EA) property, one of  $A(X), TH(X), B(X), SR(X)$  is a closed

subset of  $X$  and the pairs  $\{B, TH\}$  and  $\{A, SR\}$  are weakly compatible.

**Corollary 2.5** Let  $(X, M, *)$  be a fuzzy metric space, where  $*$  is any continuous  $t$ -norm. Let  $A, B, R, S, H$  and  $T$  be mappings from  $X$  into itself and there exists a constant  $k \in (0, 1/2)$  such that

$$M(Ax, By, kt) \geq \phi (M(SRx, THy, t), M(Ax, SRx, t), M(By, THy, t),$$

$$M(Ax, THy, \alpha t), M(By, SRx, (2 - \alpha t)))$$

for all  $x, y \in X$ ,  $\alpha \in (0, 2)$ ,  $t > 0$  and  $\phi \in \varphi$ .

Then  $A, B, R, S, H$  and  $T$  have a unique common fixed point in  $X$  provided the pair  $\{A, SR\}$  and  $\{B, TH\}$  satisfy common (EA) property,  $TH(X)$ , and  $SR(X)$  are closed subsets of  $X$  and the pairs  $\{B, TH\}$  and  $\{A, SR\}$  are weakly compatible.

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