Bayesian Estimation and Interval Estimation in an $M/\ M/\infty$ Queue

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Research Article

Abstract: A consistent estimator and a Bayes estimator of traffic intensity in an $(M/\ M/\infty):(GD/\ \infty/\infty)$ queueing model based on the number of customers present at several sampled time points are obtained. Further, consistent asymptotically normal (CAN) estimator and asymptotic confidence limits for the average number of customers in the system are obtained.

Keywords: Bayes estimator, CAN estimator, maximum likelihood estimator, $M/\ M/\infty$ queue, multivariate central limit theorem, Slutsky theorem.

1. Introduction

Most of the studies on queueing models are confined to only obtaining expressions for transient or stationary (steady state) solutions and do not consider the associated statistical inference problems. Parametric estimation, interval estimation and Bayes estimation are some of the essential tools to understand any random phenomena using stochastic models. Analysis of queueing systems in all these directions has not received much attention in the past. Whenever the systems are fully observable in terms of their basic random components such as interarrival times and service times, standard parametric techniques of statistical theory are quite appropriate. An important aspect of queueing theory is to estimate queueing parameters for which both classical and Bayesian approaches are useful. Table 1 indicates the present state of work of queueing systems, wherein both classical and Bayesian approaches are used for the estimation of queueing parameters.

Table 1: Present state of work queueing systems

Sr. No.	System Description	Authors	Estimators
1	M/M/1	Clarke (1957)	MLEs of λ and μ
2	M/M/1	Muddapar (1972)	Bayes estimators of λ and μ
3	M/M/1/∞ and M/M/1/N	Yadavalli et al (2004)	MLE of W_Q
4	M/M/c/∞ and M/M/c/N	Yadavalli et al (2006)	MLE of W_Q
5	Tandem queue with blocking and dependent structure for service times	Chandrasekhar et al (2006)	Moment estimators of L_s and W_s
6	Estimation of the Parameters of a two-phase Tandom queue with a second optional service	Ghorbani-Mandolakani et al (2013)	MLEs of the parameters

Thiruvaiyaru and Basawa(1992) adopted an empirical Bayesian approach to estimate the parameters of various queueing systems, where they used arrival and service times as the observed data. In all these models considered so far, it may be noted that MLE and Bayes estimators of queueing parameters are obtained by observing mainly number of arrivals and the number of service completions, waiting time or sojourn time in the continuous setup. But in a real life situation, it is easy to observe the number of customers at different time points. Mukherjee and Chowdhury (2005) have obtained MLE and Bayes estimator of traffic intensity in M/M/1 queueing model based on the number of customers present at several sampled time points. Recently, Paul R. Savariappan et al (2012) have obtained MLE and Bayes estimator of the parameter p based on the number of observations present at several sampled time points assuming that the stationary distribution in an M/M/1 balking situation is Negative Binomial. A two-phase tandem queueing model with a second optional service is dealt with in Mehrzad Ghorbani-Mandalakani et al (2013) and they have estimated the parameters of the model, traffic intensity and mean system size, in the steady state, via maximum likelihood and Bayesian methods. An attempt is made in this paper to obtain a consistent estimator of traffic intensity ρ in the steady state of M/M/∞ queue with unlimited service (i.e., an infinite number of servers available) based on the number of customers present at several sampled time points. Further, Bayes estimator of ρ under the same set up and minimum Bayes risk are obtained. Also MLE of the expected number of customers in the system, CAN estimator and asymptotic confidence limits for the average number of customers in the system are obtained.

2. Consistency of ρ

The density functions for interarrival times and service times are given by

$$a(t) = \lambda e^{-\lambda t} \text{ and } b(t) = e^{-\mu t}, \quad \lambda, \mu > 0, 0 < t < \infty \tag{2.1}$$

where $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ are the mean interarrival time and mean service time respectively. We assume that interarrival and service times are independently distributed. It can be shown that the steady state probability distribution of the number of customers r present in an M/M/∞ queueing system is Poisson and is given by

$$p_r = \frac{e^{-\rho}\rho^r}{r!}, \quad r = 0,1,2,3,\ldots \tag{2.2}$$

where $\rho = \frac{\lambda}{\mu} > 0$

Let $(x_1,x_2,\ldots,x_n)$ denote the number of customers present at different sampled time points $(t_1,t_2,\ldots,t_n)$. The likelihood function of the number of customers $(x_1,x_2,\ldots,x_n)$ at $(t_1,t_2,\ldots,t_n)$ is given by

$$L(\rho|x_1,x_2,\ldots,x_n) = \frac{e^{-n\rho}\rho^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \tag{2.3}$$

Now,

$\frac{\partial \log L}{\partial \rho} = 0$ implies that

$\hat{\rho} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{Y}{n}$ (say).

It readily follows that, E(Y)= nρ and Var(Y)= nρ, where $Y = \sum_{i=1}^{n} X_i$.

Since $\hat{\rho}$ is a one-to-one function of Y, it is clear that y=0,1,2,3… with the probability mass function given by

$$P_r\left[\frac{Y}{n} = u\right] = P_r[Y = nu]$$

$$= \frac{e^{-n\rho}(n\rho)^{nu}}{(nu)!} = f(y,\rho) \text{ (say)}. \tag{2.4}$$

Further,

$E[\hat{\rho}] = E\left[\frac{Y}{n}\right] = \rho$ and

$var(\hat{\rho}) = \frac{\rho}{n} \to 0 \text{ as } n \to \infty$, which shows that $\hat{\rho}$ is a consistent estimator of ρ.

By letting, $T = \frac{\hat{\rho} - E[\hat{\rho}]}{\sqrt{var(\hat{\rho})}} = \frac{\hat{\rho}-\rho}{\sqrt{\frac{\rho}{n}}}$, it can be shown that

$M_T(t) = E[e^{Tt}] \to e^{\frac{t^2}{2}} \text{ as } n \to \infty.$

In other words, $T \xrightarrow{d} N(0,1) \text{ as } n \to \infty$. (2.5)

In the next section, Bayes estimator of **ρ** and its Bayes risk are obtained by using the same data of Section 2, namely the number of customers present at several sampled time points.

3. Bayes estimator of ρ

A two parameter Gamma distribution is taken as the natural conjugate prior density for ρ. Assume that, ρ has a prior distribution Gamma with the parameters α_1 and α_2 namely

$$\tau(\rho|\alpha_1,\alpha_2) = \frac{\alpha_2^{\alpha_1}}{\Gamma\alpha_1} e^{-\alpha_2\rho}\rho^{\alpha_1-1}, \quad 0 < \rho < \infty, \quad \alpha_1,\alpha_2 > 0 \tag{3.1}$$

Note that, α_1 and α_2 are assigned known constants. The marginal pdf of Y, which is called the predictive pdf is given by

$$f^*(y) = \int_0^\infty f(y;\rho)\tau(\rho|\alpha_1,\alpha_2)d\rho$$

$$= \frac{\alpha_2^{\alpha_1}}{\Gamma(\alpha_1)} \frac{n^y\Gamma(y+\alpha_1)}{y!(n+\alpha_2)^{y+\alpha_1}} .$$

Hence, the posterior distribution of **ρ** is given by

$$q(\rho|x_1,x_2,\ldots,x_n) = \frac{f(y;\rho)\tau(\rho|\alpha_1,\alpha_2)}{\int_0^\infty f(y;\rho)\tau(\rho|\alpha_1,\alpha_2)d\rho}$$

$$= \frac{(n+\alpha_2)^{y+\alpha_1}}{\Gamma(y+\alpha_1)} e^{-(n+\alpha_2)\rho}\rho^{(y+\alpha_1)-1},$$

$$0 < \rho < \infty \tag{3.2}$$

Remark 3.1 The posterior distribution of ρ is also the Gamma distribution with the parameters $(n+\alpha_2, y+\alpha_1)$.

Remark 3.2 The posterior pdf of ρ reflects both prior information (α_1,α_2) and the sample information $y = \sum_{i=1}^{n} x_i$. Thus, the Bayes estimator of ρ under the squared error loss is given by

$$E(\rho|x_1,x_2,\ldots,x_n) = \int_0^\infty \rho\, q(\rho|x_1,x_2,\ldots,x_n)\, d\rho$$

$$= \frac{(y+\alpha_1)}{(n+\alpha_2)} \tag{3.3}$$

It may be noted that

$$E(\rho|x_1,x_2,\dots,x_n) = \frac{(y+\alpha_1)}{(n+\alpha_2)} = \left(\frac{n}{n+\alpha_2}\right)\left(\frac{y}{n}\right)\left(\frac{\alpha_2}{n+\alpha_2}\right)\left(\frac{\alpha_1}{\alpha_2}\right),$$

which is the weighted average of the maximum likelihood estimator $\frac{y}{n}$ and the mean $\frac{\alpha_1}{\alpha_2}$ of the prior pdf of the parameter ρ, where the respective weights are $\frac{n}{n+\alpha_2}$ and $\frac{\alpha_2}{n+\alpha_2}$. Further, the minimum posterior risk associated with this Bayes estimator is given by $V_\rho(\hat{\rho}^B|x_1,x_2,\dots,x_n) = E[\hat{\rho}-\rho]^2$

$$= \int_0^\infty (\hat{\rho}-\rho)^2 q(\rho|x_1,x_2,\dots,x_n)\,d\rho$$

$$= \frac{(y+\alpha_1)}{(n+\alpha_2)^2} \qquad (3.4)$$

Now, the minimum Bayes risk of $\hat{\rho}^B$ is given by $E[V_\rho(\hat{\rho}^B|x_1,x_2,\dots,x_n)]$ with respect to the marginal distribution $h(x_1,x_2,\dots,x_n)$ of $(X_1,X_2,\dots,X_n)$.

Hence,

$$h(x_1,x_2,\dots,x_n) = \int_0^\infty L(\rho|x_1,x_2,\dots,x_n)\tau(\rho|\alpha_1,\alpha_2)\,d\rho = \frac{\alpha_2^{\alpha_1}}{\alpha_1\prod_{i=1}^{n}x_i!}\frac{\Gamma(y+\alpha_1)}{(n+\alpha_2)^{y+\alpha_1}}$$

Now, the minimum Bayes risk $r_{\tau,\hat{\rho}^B}$ of $\hat{\rho}^B$ is given by

$$r_{\tau,\hat{\rho}^B} = E[V_\rho(\hat{\rho}^B|x_1,x_2,\dots,x_n)] = E\left[\frac{(y+\alpha_1)}{(n+\alpha_2)^2}\right]$$

$$= \sum_{y=0}^{\infty}\frac{(y+\alpha_1)}{(n+\alpha_2)^2}h(x_1,x_2,\dots,x_n)$$

$$= \frac{\alpha_2^{\alpha_1}}{(n+\alpha_2)^{\alpha_1+2}\Gamma(\alpha_1)}\sum_{y=0}^{\infty}\frac{(\alpha_1+y)\Gamma(\alpha_1+y)}{\prod_{i=1}^{n}x_i!(\alpha_2+n)^y}$$

4. Statistical inference for an $M/M/\infty$ queue

In this section, the maximum likelihood estimator for the expected number of customers in the system, CAN estimator and a 100(1-α)% confidence interval for the expected number of customers in the system are obtained. A numerical example is also given.

4.1 Maximum likelihood estimator for the expected number of customers in the system

From the Poisson distribution, the expected number of customer in the system given by

$$L_S = \sum_{r=0}^{\infty} r p_r = \sum_{r=0}^{\infty} r\frac{e^{-\rho}\rho^r}{r!} = \rho = \frac{\lambda}{\mu} \qquad (4.1)$$

Let $X_1,X_2,\dots,X_n$ and $Y_1,Y_2,\dots,Y_n$ be two random samples each of size n drawn from an exponential interarrival time population and exponential service time population with the parameters λ and μ respectively. It is clear that, $E(\bar{X}) = \frac{1}{\lambda}$ and $E(\bar{Y}) = \frac{1}{\mu}$, where $\bar{X}$ and $\bar{Y}$ are the sample means of interarrival times and service times, respectively.

Let $\theta_1 = \frac{1}{\lambda}$ and $\theta_2 = \frac{1}{\mu}$ respectively. Clearly the expected number of customers in the system given in (4.1) reduces to $L_S = \frac{\theta_2}{\theta_1}$ and hence the MLE of L_S (using the invariance property of maximum likelihood estimators) is given by

$$\hat{L}_S = \frac{\bar{Y}}{\bar{X}} \qquad (4.2)$$

It may be noted that $\hat{L}_S$ given in (4.2) is real valued function in $\bar{X}$ and $\bar{Y}$, which is also differentiable. Consider the following application of multivariate central limit theorem, see Radhakrishna Rao (1974).

4.2 Application of multivariate central limit theorem

Suppose $T_1', T_2', T_3', \dots$ are independent and identically distributed k- dimensional random variables such that $T_n' = (T_{1n}, T_{2n}, T_{3n}, \dots, T_{kn})$, n = 1,2,3,… having the first and second order moments $E(T_n) = \mu$ and $var(T_n) = \Sigma$. Define the sequence of random variables

$$\bar{T}_n = (\bar{T}_{1n}, \bar{T}_{2n}, \dots, \bar{T}_{kn}),\ n = 1,2,3\dots,$$

where $\bar{T}_{in} = \frac{1}{n}\sum_{j=1}^{n} T_{ij},\ i = 1,2,3,\dots,k$.

Further, $\sqrt{n}(\bar{T}_n - \mu) \xrightarrow{d} N(0,\Sigma)$ as $n \to \infty$.

4.3 CAN Estimator

By applying the multivariate central limit theorem given in Section 4.2, it can be readily seen that $\sqrt{n}[(\bar{X},\bar{Y}) - (\theta_1,\theta_2)] \xrightarrow{d} N(0,\Sigma)$ as $n\to\infty$, where the dispersion matrix $\Sigma = ((\sigma_{ij}))$ is given by $\Sigma = \mathrm{diag}(\theta_1^2,\theta_2^2)$. Again from Radhakrishna Rao (1974), we have $\sqrt{n}(\hat{L}_S - L_S) \xrightarrow{d} N(0,\sigma^2(\theta))$ as $n\to\infty$,

where

$$\sigma^2(\theta) = \sum_{i=1}^{2}\left(\frac{\partial L_S}{\partial\theta_i}\right)^2\sigma_{ii} = \frac{2\theta_2^2}{\theta_1^2}. \qquad (4.3)$$

Thus $\hat{L}_S$ is a CAN estimator of L_S. There are several methods for generating CAN estimators and the method of moments and the method of maximum likelihood are commonly used to generate such estimators, see Sinha (1986).

4.4 Confidence Interval for the Expected Number of Customers in the System

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by the consistent estimator $\hat{\theta}$ namely $(\bar{X}, \bar{Y})$. Let $\hat{\sigma}^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of $\theta, \hat{\sigma}^2$ is consistent estimator of $\sigma^2(\theta)$. That is, $\hat{\sigma}^2 \xrightarrow{P} \sigma^2(\theta) \, as \, n \to \infty$.

By Slutsky theorem

$$(X_n \xrightarrow{d} X, Y_n \xrightarrow{P} b \Rightarrow \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{b}, b \neq 0)$$

, we have

$$\frac{\sqrt{n}(\hat{L}_S - L_S)}{\hat{\sigma}} \xrightarrow{d} N(0,1).$$

That is, $Pr\left[-k_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{L}_S - L_S)}{\hat{\sigma}} < k_{\frac{\alpha}{2}}\right] = (1-\alpha),$ where $k_{\frac{\alpha}{2}}$ is obtained from normal tables. Hence, 100(1-α)% asymptotic confidence interval for L_S is given by

$$\hat{L}_S \pm k_{\frac{\alpha}{2}} \frac{\hat{\sigma}}{\sqrt{n}} \qquad (4.4)$$

In the following section 4.5, we provide a numerical illustration to study the performance of $\hat{L}_S$.

4.5 Numerical illustration

The CAN estimator of L_s is obtained and its performance is studied. Here 5000 random samples are generated independently 50 times from the exponential distributions assuming λ=2 and μ=3. The CAN estimator of L_s namely $\hat{L}_s$ is obtained using these estimates. Table 2 shows the calculated values of L_s , $\hat{L}_s$, $\hat{\sigma}_{ii}$, 95 % lower and upper confidence limits and the differences between L_s and $\hat{L}_s$, which is the bias of the estimate $\hat{L}_s$. The Mean Square Error is obtained by using the formula $\frac{1}{50}\sum(\hat{L}_s - L_s)^2$ to find the performance of CAN estimator of L_s. The Mean Square Error is 0.000179995 and it is found to be very close to zero. Hence we conclude that the proposed estimator performs reasonably well.

Table 2: Values of $\hat{L}_S$ and 95 % confidence limits of L_S.

Sl. No	$\hat{L}_S$	Lower limit	Upper limit
1	0.6683	0.4207	0.9159
2	0.6459	0.4146	0.8771
3	0.6598	0.4185	0.9012
4	0.6760	0.4227	0.9293
5	0.6558	0.4174	0.8943
6	0.6524	0.4165	0.8884
7	0.6639	0.4196	0.9083
8	0.6411	0.4133	0.8690
9	0.6723	0.4217	0.9228
10	0.6611	0.4188	0.9034
11	0.6554	0.4173	0.8935
12	0.6696	0.4210	0.9181
13	0.6742	0.4222	0.9262
14	0.6758	0.4226	0.9291
15	0.6568	0.4177	0.8960
16	0.6557	0.4173	0.8940
17	0.6684	0.4207	0.9160
18	0.6898	0.4260	0.9535
19	0.6888	0.4258	0.9519
20	0.6776	0.4231	0.9322
21	0.6382	0.4124	0.8640
22	0.6742	0.4222	0.9262
23	0.6578	0.4179	0.8978
24	0.6548	0.4171	0.8924
25	0.6759	0.4226	0.9291
26	0.6732	0.4220	0.9245
27	0.6612	0.4188	0.9036
28	0.6568	0.4176	0.8959
29	0.6483	0.4153	0.8813
30	0.6767	0.4228	0.9305
31	0.6416	0.4134	0.8699
32	0.6795	0.4235	0.9355
33	0.6795	0.4235	0.9355
34	0.6514	0.4162	0.8867
35	0.6396	0.4128	0.8664
36	0.6753	0.4225	0.9281
37	0.6815	0.4240	0.9390
38	0.6625	0.4192	0.9059
39	0.6698	0.4211	0.9185
40	0.6665	0.4202	0.9127
41	0.6674	0.4205	0.9143
42	0.6631	0.4193	0.9069
43	0.6505	0.4159	0.8851
44	0.6956	0.4274	0.9638
45	0.6722	0.4217	0.9228
46	0.6551	0.4172	0.8930
47	0.6721	0.4217	0.9225
48	0.6712	0.4215	0.9210
49	0.6770	0.4229	0.9310
50	0.6838	0.4246	0.9430
Mean Square Error		**0.00018**	

Conclusion

A consistent estimator and Bayes estimator of traffic intensity ρ of the queueing model M/M/∞ based on the number of customers present at several sampled time points are obtained. Also, CAN estimator and asymptotic confidence limits for the expected number of customers in the system are obtained. Further, simulation study is carried out to obtain the mean square error to assess the performance of CAN estimator of L_s and concluded that the proposed estimator of the expected number of customers in the system $\hat{L}_s$ performed reasonably well.

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