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Optimization of Time Varying Investment Returns of Insurance Company under Power Utility Function

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Research Article

Abstract: In this study, we take the risk reserve of an insurance company to follow Brownian motion with drift and tackle an optimal portfolio selection problem of the company. The investment case considered was insurance company that trades two assets: the money market account (bond) growing at a rate r_t that is linear function of time $(r_t = \varepsilon + \sigma t)$ anda risky stock with an investment behavior in the presence of a stochastic cash flow or a risk process, continuously in the economy. Our focus was on obtaining investment strategies that are optimal in the sense of optimizing the returns of the company. We established among others that, the optimized investment in the assetsand the optimal value function are dependent onhorizon and the wealth.It is recommended that the managers of the assets of the insurance company should take into consideration this horizon dependency when making policy decisions.

Key words: Stochastic optimal control, Investment returns, Power Utility function, Brownian motion process.

Introduction

Optimal portfolio selection problem is of practical importance in finance and insurance Mathematics. Earlier work in this area can be traced back to Markowitz's mean variance model (Markowitz 1959). Samuelson (1969) extended the work of Markowitz to a dynamic set up. He used a dynamic stochastic programming approach; he succeeded in obtaining the optimal decision for consumption investment model. Merton (1971) used the stochastic optimal control methods in continuous finance to obtain a closed form solution to the problem of optimal portfolio strategy under specific assumptions about the asset returns and the investor preferences. These days, insurance companies invest both in the money market and stocks. Due to the high risks involved in the stock market, investment strategies and risk management are becoming more important. Using the Hamilton-Jacobi-Bellman equation, Hipp and Plum (2000), determined the strategy of investment which minimizes the probability of ruin modeling the price of the stocks by geometric Brownian motion. Under the same hypothesis, Gaieret. al., (2003) obtained an exponential bound with a rate that improves the classical Lundberg parameter. The optimal trading strategy they found involved investing in the stock a constant amount of money independent of the reserve. This work used Ferguson (1965) approach which conjectured that maximizing the exponential utility of terminal wealth is intrinsically related to minimizing the probability of ruin. Browne (1995) verified the conjecture in a model without interest rate where the stock followed a geometric Brownian motion and the risk process of an insurance company being Brownian motion with drift. In this work consider herein the wealth process of the reserve of an insurance company, modeling the risk process as Brownian motion with drift when there is investment in a risk free asset that is a linear function of time and a risky asset (stock) that followed a geometric Brownian motion. We find an optimal investment strategy for the company that maximizes power utility function where the risk-free asset has a time-varying rate of return. This investment strategy will decide how the company should invest between the risk-free bond and a risky stock subject to its obligation to pay policy holders when claims occur. The optimal investment problem consists of finding the portfolio strategy that maximizes the expected value of the utility function at some predetermined moment in future. A large number of researches have been and are being done in both finance and insurance mathematics on the problem of optimal portfolio allocation/selection. Browne (1999) considered a portfolio problem in continuous time where the objective of the investor or money manager was to exceed the performance of a given stochastic benchmark, as is often the case of institutional money management. Unlike in our paper where the benchmark is a fixed point, the benchmark in his paper was a stochastic process that needed not be a perfectly correlated with the investment opportunities and the market is in a sense incomplete. He solved a variety of investment problems related to the achievement of goals for example he finds an optimal investment strategy that maximizes the probability that the return of the investor's portfolio beats the return of the benchmark by a given percentage. He also considered objectives related to the minimization of the expected time until the investor beats the benchmark. The problem of maximizing the expected discounted reward of outperforming the benchmark as well as minimizing the discounted penalty paid upon being outperformed was discussed.

Castillo and Parrocha (2008) unlike in our model, considered an insurance business with a fixed amount available for investment in a portfolio consisting of one non-risky asset and one risky asset. They presented the Hamilton-Jacobi-Bellman (HJB) equation and demonstrated its use in finding the optimal investment strategy based on some given criteria. The objective of the resulting control problem was to determine the investment strategy that minimized infinite ruin probability. The existence of a solution to the resulting HJB equation was then shown by verification theorem. A numerical algorithm is also given for analysis. Hipp and Plum (2000) modeled the risk process of an insurance company as a compound Poison process unlike ours where the risk process is modeled by Brownian motion with drift. In their paper, they applied stochastic control to answer the following question: if an insurer has the possibility of investing part of his surplus into a risky asset, what is the optimal strategy to minimize the probability of ruin. They observed that the probability of ruin of the risk process can be minimized by a suitable choice of an investment strategy by a capital market index. The optimal strategy was computed using the Bellman equation. They also proved the existence of a smooth solution and a verification theorem, and gave explicit solutions in some cases with exponential claim distributions as well as numerical results in a case with Pareto claim size distribution. It was observed that the optimal amount invested cannot be bounded for that last case. Begona, et al. (2006) studied an optimal investment problem of an insurer when the company has the opportunity to invest in a risky asset using stochastic control techniques. They observed that closed form solutions to the problem could only be obtained when the investor's risk preferences are exponential and as well as an estimate of the ruin probability was possible when optimal strategy was used.

Promislow and Young (2005) extended the work of Browne (1995) and Schmidli (2001), in which they minimized the probability of ruin of an insurer facing a claim process modeled by a Brownian motion with drift. They consider two controls to minimize the probability of ruin;

1. Investing in a risky asset (constrained and the non-constrained cases)
2. Purchasing quota-share reinsurance.

They obtained an analytic expression for the minimum probability of ruin and their corresponding optimal controls. They also demonstrated their results with numerical results.

Bayraktar and Young (2008) worked on a problem involving individual consumers and especially beneficiaries of endowment funds who generally employ strategies such that consumption never decreases (ratcheted) or at least they try to do this. They assumed that an agent's rate of consumption is ratcheted; that is it forms a non-decreasing process. They assumed that the agent invests in a financial market with one risk less asset and one risky asset with the latter's price following geometric Brownian motion as in Black Scholes model. Given the rate of consumption of the agent, they act as financial advisers and find optimal investment strategies for the agent who wishes to minimize his/her probability of running out of money either before dying or before the organization holding the endowment fails due to causes other than the ruin of the fund itself. They solved this minimization problem using stochastic optimal control techniques.

Pliska (1986) considered the problem of choosing a portfolio of securities so as to maximize the expected utility of wealth at terminal planning horizon which he solved through stochastic calculus and convex analysis. He decomposed his problem into two: security prices modeled as semi martingales and trading strategies modeled as predictable processes. The set of terminal wealth is identified as a subspace of integrable random variables. He used convex analysis to derive necessary and sufficient conditions for optimality and existence result. In his second sub problem, admissible strategies that generate the optimal terminal wealth as a martingale representation problem had to be found. This approach has a primary advantage in that explicit formula can readily be derived for optimal terminal wealth and corresponding expected utility. Liu and Yang (2004) studied optimal investment strategies of an insurance company. They assume that an insurance company receives premiums at a constant rate, the total claims are modeled by a compound Poison process, and the insurance company can invest in the money market (bonds) and in a risky asset such as stocks. Their model generalizes that in the Hipp and Plum (2000) by including a risky free asset. The investment behavior in this paper was investigated numerically for various claim size distributions. The optimal policy and the solution of the associated Hamilton Jacobi Bellman equation are then computed under each assumed distribution. The effects of the changes in the various factors such as the stock volatility, on optimal investment strategies and survival probability are investigated. They further generalize to cases in which borrowing constraints or reinsurance is present. Laubis and Jeng-Eng (2008) considered an optimal asset allocation problem of an insurance company. In their model, an insurance company is represented by a compound Poison risk process which is perturbed by diffusion and has investments. The investments are in both risky and risk-free types of assets similar to stocks/real estates and bonds respectively. The insurance company can borrow at a constant interest rate in the event of a negative surplus. Numerical analysis appears to show that an optimal asset allocation range can be estimated for certain parameters and can be compared with using insurance data. Using a conservative method to minimize the probability of ruin, they were able to show that a reasonable optimal asset allocation range for a typical insurance is about 4.5 to 8 percent invested in risky stock/real estate assets. An inequality and the exact solution are obtained for the pure diffusion equation. In addition the asymptotic form of the ruin probability is shown to be a power function. Oksendal and Sulem (2002) investigated a market with one risk-free and one risky asset in which the dynamics of the risky asset are governed by a geometric Brownian motion. They considered an investor who consumes from a bank account and has the opportunity at any time to transfer funds between two assets, and assumes that these transfers involve a _xed transaction cost which was independent of the size of the transaction plus its cost proportional to the size of transaction. Their objective was to maximize the cumulative expected utility of consumption over planning horizon and they formulated the problem as a combined stochastic control/ impulse control.

Merton (1969, 1971, and 1996) constructed and analyzed optimal continuous-time allocation problems under uncertainty. He considered a model in which the prices of the risky assets are generated by correlated geometric Brownian motions and assumes that the portfolio can be rebalanced instantly and with no cost. His main objective in this work is to maximize the net expected utility of consumption plus the utility of terminal wealth. In order to keep the proportions invested in the risky asset equal to a constant vector and to consume at a rate proportional to the total wealth, he considers an optimal trading strategy which consists of an infinite number of transactions and the utility functions in the constant relative risk aversion (CRRA) case. Qian and Lin (2009) considered an insurance company whose surplus (reserve) is modeled by a jump diffusion risk process. The insurance company can invest part of its surplus in n risky assets and purchase a proportional reinsurance for claims. Their main goal is to find an optimal investment and proportional reinsurance policy which minimizes ruin probability. They apply stochastic control theory to solve the problem. They obtained closed form expression for the minimum probability, optimal investment and proportional reinsurance policy. They found out that the minimum ruin probability satisfies the Lundberg equality. They also investigated the diffusion volatility parameter, the market price of risk and the correlation coefficient on the minimal ruin probability, optimal investment and proportional reinsurance policy through numerical calculations. Azcue and Muler, (2009) considered that the reserve of an insurance company follows a Cramer-Lundberg process. They considered that the management of an insurance company had the possibility of investing part of the reserve in a risky asset. They considered that the risky asset was a stock as it is with most of the rest of the studies whose price process was a geometric Brownian motion. Their main aim was to find a dynamic choice of investment policy which would minimize the ruin probability of the insurance company. They characterized the optimal value function as the classical solution of the associated Hamilton-Jacobi-Bellman equation which was a non linear second order integro-differential equation. Numerical solutions were obtained for comparison with the results of the unconstrained cases that were studied earlier by Hipp and Plum (2000). Their study revealed that the optimal strategies of the constrained and unconstrained problems do not coincide. Kostadinova (2007) considered a stochastic model for the wealth of an insurance company which has the possibility to invest into a risky asset and a risk-less asset under constant mix strategy. This total claim amount being modeled by compound Poison process and the price of the risky asset considered to follow a general Levy process. They investigated the resulting integrated risk process and the corresponding discounted net loss process. This opened up a way to measure the risk of a negative outcome of the integrated risk process in a stationary way.

They provided an approximation of the optimal investment strategy that maximizes the expected wealth of the insurance company under the risk constraint on the Value-at-Risk. Grandits and Gaier (2002) studied the infinite ruin probability problem in the classical Cramer-Lundberg model, where the company was allowed to invest their money in a stock whose price followed geometric Brownian motion. Starting from an integro-differential equation for the maximal survival probability, they analyzed the case of claim sizes which have distribution functions F with regularly varying tails. Their results showed that if $1 - F$ is regularly varying with index $\rho < -1$, then the ruin probability is also regularly varying with same index $\rho < -1$and this was under an assumption of zero interest rates. Bai and Liu (2007) considered a classical risk process model and allowed investment into a risk-free asset as well as proportional reinsurance. The optimal proportional reinsurance strategy was found to minimize the probability of ruin of an insurance company. The problem was treated under two cases: The first case was trivial, the corresponding probability of ruin and the optimal proportional reinsurance strategy were obtained directly. The second case, firstly the existence of the solution to the Hamilton-Jacobi-Bellman (HJB) equation was provided. Then the minimal probability of ruin and the optimal proportional reinsurance strategy were obtained by a verification theorem. Liu,Bai and Yiu (2012) considered a constrained investment problem with the objective of minimizing the ruin probability. In their paper, they formulated the cash reserve and investment model for the insurance company and analyzed the Value-at-Risk (VaR) in a short time horizon. For risk regulation, they imposed it as a risk constraint dynamically. Then the problem was therefore to minimize the probability of ruin together with the imposed risk constraint. By solving the corresponding Hamilton-Jacobi-Bellman equations, they derived analytic expressions for the optimal value function and the corresponding optimal strategies. Looking at the Value-at-Risk alone, they were able to show that it was possible to reduce the overall risk by an increased exposure to the risky assets with the stochastic of the fundamental insurance business. Studying the optimal strategies, they found out that a different investment strategy would be in place depending on the Sharpe ratio of the risky asset. Luo (2008) considered an optimal dynamic control problem for an insurance company with opportunities of proportional reinsurance and investment. The company can purchase proportional reinsurance to reduce the risk level and invest its surplus in a financial market that a Black-Schole risky asset and a risk-free asset. Unlike in our model, when investing in the risk-free asset, three practical borrowing constraints are studied individually :(B1) the borrowing rate is higher than the lending (saving) rate, (B2) the dollar amount borrowed is no more than $k > 0$, where k is a fixed limit, and finally (B3) the proportion of the borrowed amount to the surplus level is no more than k > 0. Under each of the constraints, the objective is to minimize the probability of ruin. Classical stochastic control theory is applied to solve the problem. Specifically, the minimal ruin probability functions are obtained in closed form by solving Hamilton-Jacobi-Bellman (HJB) equations, and other associated optimal reinsurance investment policies are found by verification theorem. From the discussions above, we note that the risk process of an insurance company in most of the papers is modeled by the Cramer Lundberg model while investment is done with either two assets or with a single asset and reinsurance. This work aims at optimization of investment returns of the insurance company, build a foundation for further research and also contribute to improved insurance business.

2. Statement of the problem and the Formulation

Most of the portfolio selection studies in insurance mathematics have focused on finding optimal investment strategies that minimize the probability of ruin when the risk process of an insurance company follows the Cramer-Lundberg model, however this does not come as easily as it presents difficult numerical computations of the ruin probability. In this work therefore, we intend to study the portfolio selection problem when the risk process of an insurance company follows Brownian motion with drift. We take this approach because we can obtain analytical solutions with less difficulty. The company is expected to invest in two assets that is a risk-free bond having time-varying rate of return and a risky asset whose price dynamics follows geometric Brownian motion, which is quite different from the work of Browne (1995) who considered only one investment opportunity and that of Wokiyi (2012) where the risk free asset has constant rate of return. We are interested in obtaining optimal investment strategies that would help an investor decide how much of his wealth he can invest in each asset so as to optimize the expected utility function of the company.

2.1. Weak Convergence of Risk process to Brownian motion.

This section will be devoted to the Brownian motion approximation in Risk theory and will be based on the work of Iglehart (1969). He assumed that the distribution of the claim sizes belong to the domain of attraction of the normal law. Thus, such claims attain big values with small probabilities. This assumption will cover many practical situations in which the claim size distributions possesses a finite second moment and claims constitute an identically independent sequence. The claims counting process does not have to be independent of the sequence of the claim sizes as it is assumed in many risk models and in general can be a renewal process constructed from random variables having a finite moment.

To approximate the risk process of an insurance company by Brownian motion, consider a sequence of risk process $R_n(t)$ defined in the following way;

$$R_n(t) = u_n + c_n t - \sum_{i=1}^{N(t)} Y_k^{(n)} \quad (1)$$

where u_n is the initial risk reserve of the insurance company, c_n is the gross risk premium per unit time paid by the policy holders and the sequence $\{Y_k^{(n)}: k \in N\}$,describes the consecutive claim sizes. Assume also that $E\left(Y_k^{(n)}\right) = \mu_n$ and $var\left(Y_k^{(n)}\right) = \sigma_n^2$.

The point process $N = \{N(t) : t \geq 0\}$ counts claims appearing up to time t , that is

$$N(t) = \max\{k: \sum_{i=1}^{k} T_i \leq t\} \quad (2)$$

where$\{T_k: k \in N\}$ is an identically independent sequence of non negative random variables describing the times between the arriving claims with $E(\{T_k) = \frac{1}{\lambda} \geq 0$.

If T_k are exponentially distributed then $N(t)$ is a Poison process with intensity λ.

Theorem 1: If $u_n = un^{\frac{1}{2}} + o(n^{\frac{1}{2}})$,$c_n = cn^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$,$\mu_n = \mu n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$ $\sigma_n{}^2 \to \sigma^2andE[\left(Y_i{}^{(n)}\right)^{2+\epsilon}]$ is bounded in n for some $\epsilon > 0$, Then the sequence of classical reserve processes converges weakly to a stochastic process of the form;

$$R_t = u + \Gamma + \sigma\lambda^{\frac{1}{2}} B_t^{(1)} \quad (3)$$

Where $\Gamma = (\Gamma_t)_{t\geq 0}$ with $\Gamma_t = (c - \lambda\mu)t$ and $(B_t^{(1)})_{t\geq 0}$ is a standard Brownian motion (see Iglehart ,1969,for proof).

2.2. The Model

Adapting the formulation of Osu and Ihedioha (2012), we assume that insurance company trades two assets continuously in the economy. The first asset is the money market account (bond) growing at a rate r_t that is linear function of time ($r_t = \varepsilon + \sigma t$), instead of a constant as in Wokiyi (2012).$r_t = \varepsilon + \sigma t$, *($\alpha > 0, 0 \leq \sigma \leq 1$) is a decreasing (or an increasing) linear* function of t *as* $t \to \infty$ .The parameterε is the initial investment on the money market account which determines the speed of a mean-reversion to the stationary level. $\sigma$is the acceleration coefficient which is the volatility (variance) of the process and is proportional to the level of the interest rate. It decreases as the interest rate$r_t \to 0$. The equation governing the dynamics of the money market account (bond) is given as;

$$dB_t = (\varepsilon + \sigma t)B_t dt. \quad (4)$$

We assume that there is only one risky stock available for investment (e.g a mutual fund) whose price at any time t will be denoted by S_t. We will also assume that the price process of the risky stock follows the geometric Brownian motion:

$$dS_t = S_t dZ_t , \quad (5)$$

whereZ_t is a Brownian motion with drift $\mu$and diffusion parameter σ, that is, $dZ_t = \mu dt + \sigma dB_t^{(2)}$, where μ and $\sigma$are constants and $B_t^{(2)}: t \geq 0$ is a standard Brownian motion.

In classical theory of risk, the true net claims process say $\{R_t\}$ is usually modeled as;

$$R_t = u + ct - \sum_{i=1}^{N(t)} Y_i. \quad (6)$$

where u is the initial risk reserve, c is the premium income rate per unit time, N_t is the number of claims up to time t usually modeled as a stationary renewal process with rate λ and Y_i is the size of thei^{th} claim with $\{Y_i: i \geq 1\}$assumed to be an identically independent sequence as shown in the previous section, $\alpha = c - \mu\lambda$ and $\beta^2 = \sigma^2\lambda$and these can also be written as $\alpha = c\lambda E(Y_1)$ and $\beta^2 = \lambda E({Y_1}^2)$So the parameter α can be understood as the relative safety loading of the claims process.

We are concerned with investment behavior in the presence of a stochastic cash flow or a risk process which we will denote by $R_t : t \geq 0$ which describes a Brownian motion with drift$\alpha$and diffusion parameter σ that is R_t satisfies the stochastic differential equation;

$$dR_t = \alpha dt + \beta\, dB_t^{(1)} \quad (7)$$

whereα and β are constants (with $\beta \geq 0$).

We also allow the two Brownian motions to be correlated and we denote their correlation coefficient byρthat is $E(B_t^{(1)} B_t^{(2)}) = \rho t$. We will not consider the uninteresting case of ρ^2, in which case there would be only one source of randomness in the model.

The company is allowed to invest its surplus in the risky stock and we will denote the total amount of money invested in the risky stock at time t under an investment policy π as π_t where $\{\pi_t\}$ is a suitable admissible adapted control process, that is, π_t is a non anticipative function and satisfies for any T,

$$\int_0^T {\pi_t}^2 \mathrm{d}t < \infty , \quad (8)$$

almost surely.

We assume that W_t is the total wealth of an insurance company. We also assume that the insurance company allocates its wealth as follows: Let $\pi_t$be the total amount of the company's wealth that is invested in risky assets and remaining balance ($W_t - \pi_t$) be invested in a risk-less asset (bond/market).

We note that $\pi_t$may become negative, which is to be interpreted as short selling a stock. The amount invested in the bond, $W_t - \pi_t$ may also be negative, and this amounts to borrowing at the interest rate r. For any policy π, the total wealth process of an insurance company evolves according to the stochastic differential equation as;

$$dW_t^{\pi} = \pi_t \frac{dS_t}{S_t} + (W_t - \pi_t)\frac{dB_t}{B_t} + dR_t. \quad (9)$$

Substituting the expressions for S_t,$B_t$and R_t, the stochastic differential equation for the wealth process of the company then reduces to;

$$dW_t^{\pi} = \left[w(\varepsilon + \sigma t) + \pi_t\left(\mu - (\varepsilon + \sigma t)\right) + \alpha\right]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}. \quad (10)$$

Assuming $B_t^{(1)}$ and $B_t^{(2)}$are correlated standard Brownian motions, with correlation coefficient ρ,the quadratic variation of the wealth process is;

$$d < W >_t = ({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)dt \quad (11)$$

Definition: A control process π_t is said to be admissible for an initial endowment $w \geq 0$ if the wealth process generated by the stochastic differential equation (3.10) satisfies, $W_t \geq 0; 0 \leq t \leq T$; almost surely. Then the quadratic variation of the wealth process is given by;

$$d < W >_t = dW_t^{\pi}.W_t^{\pi}$$
$$= [\mu\pi_t + (\varepsilon + \sigma t)(W_t - \pi_t) + \alpha]^2 dtdt + 2\pi_t \sigma[\mu\pi_t + (\varepsilon + \sigma t)(W_t - \pi_t) + \alpha]dtdB_t^{(2)} + 2\beta [\mu\pi_t + (\varepsilon + \sigma t)(W_t - \pi_t) + \alpha]dtdB_t^{(1)} + ({\pi_t}^2\sigma^2)dB_t^{(1)}B_t^{(2)} + 2\pi_t \sigma\beta dB_t^{(1)}B_t^{(2)} + \beta^2 B_t^{(1)}B_t^{(2)}. \quad (12)$$

But $dB_t^{(1)}dB_t^{(1)} = dB_t^{(2)}dB_t^{(2)} = dt$, $dtdB_t^{(1)} = dtdB_t^{(2)} = dtdt = 0$ and since $B_t^{(1)}$and$B_t^{(2)}$are correlated Brownian motions, with correlations coefficient ρ, then $dB_t^{(1)}dB_t^{(2)} = \rho dt$, hence the expression for the quadratic variation then reduces to (11).

If $\rho^2 \neq 1$, this model is incomplete in a very strong sense in that the random cash flow or the random endowment R_t, can not be traded on the security market, and therefore the risk to the investor cannot be eliminated under any circumstance.

We put no constraints on the control π_t except for the particular case where the possibility of borrowing is not allowed, we allow $\pi < 0$as well as $\pi_t > W_t^{\pi}$. In the first instance the company is shorting stock while in the second instance the company borrows money to invest long in the stock.

The company can always borrow money for as long as it has a positive net worth, that is, $W_t^{\pi} > 0$ and we don't allow the company to borrow money once it's bankrupt and thus the possibility of ruin is of real concern.

Suppose the investor has a power utility function, the Arrow-Pratt measure of relative risk aversion (RRA) or coefficient of relative risk averse is defined as;

$$R(w) = \frac{U''(w)}{U'(w)}, \quad (13)$$

where$w$ is the wealth level of an investor and a, k are constants. We consider a special case where the utility function is of the form,

$$U(w) = \frac{w^{1-c} - c}{1-c}, \quad (14)$$

which has a constant relative risk averse parameter c. The motivation to use power utility stems from the fact that power utility functions with a constant relative risk averse are related to survival as well as growth objectives that may be taken up by a prospective investor.

The most straight forward implication of increasing or decreasing the relative risk averse, and the ones that motivate a focus on these concepts, occur in the context of forming a portfolio with one risky asset and risk free asset. If an investor experiences an increase in wealth he will choose to increase (or keep unchanged or decrease) the fraction of the portfolio held in the risky asset if the relative risk averse is decreasing (or constant, or increasing).

The insurance company's problem can therefore be written as:

$$\left.\begin{array}{l} sup_{\pi}\{\mathcal{A}^{\pi}V(t,w)\} = 0 \\ V(T,w) = U(w) \end{array}\right\} \quad (15)$$

Where $V(t,w) = sup_{\pi}E^{(t,w)}[U(W_T^{\pi})]$ (16)

subject to:

$$dW_t^{\pi} = [w(\varepsilon + \sigma t) + \pi_t(\mu - (\varepsilon + \sigma t)) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}.$$

3. The Optimization of The Insurance Company's Returns

In this section, weoptimize the insurance company's returns under both power and exponential utility functions.

To solve the dynamic optimization problem, we derive the Hamilton-Jacobian-Bellman (HJB) partial differential equation starting with the Bellman equation:

$$V(w,t;T) = sup_{\pi} E[V(w', t + \Delta t; T)] \quad (17)$$

wherew',denotes the wealth of the company at time $t + \Delta t$. Hence,

$$sup_{\pi} E[V(w', t + \Delta t; T) - V(w,t;T)] = 0 \quad (18)$$

Dividing both sides of the equation by Δt and taking limit to zero, the Bellman equation becomes;

$$sup_{\pi} \frac{1}{dt} E[dV] = 0 \quad (19)$$

By Ito's lemma, this states that;

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial w}dw + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(dw)^2$$, Miao (2010).

Rewriting the Ito's lemma by substituting in the stochastic differential equation (SDE) for $dW$and $d < W >$,we obtain:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial w}[w(\varepsilon + \sigma t) + \pi_t(\mu - (\varepsilon + \sigma t)) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)} + \frac{\partial^2 V}{2\partial w^2}\left[[w(\varepsilon + \sigma t) + \pi_t(\mu - (\varepsilon + \sigma t)) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}\right]^2.$$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial w}[w(\varepsilon + \sigma t) + \pi_t(\mu - (\varepsilon + \sigma t)) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)} + \frac{\partial^2 V}{2\partial w^2}({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)dt. \quad (20)$$

Applying (20) to the Bellman equation (19) we get;

$$sup_{\pi}\frac{1}{dt}E\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial w}[w(\varepsilon + \sigma t) + \pi_t(\mu - (\varepsilon + \sigma t)) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)} + ({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)dt\right] = 0. \quad (21)$$

where;

$$E\left(dB_t^{(1)}\right) = E\left(dB_t^{(2)}\right) = 0, \quad (22)$$

the HJB equation is;

$$V_t + \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha\right]V_W + ({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)V_{WW} = 0, \quad (23)$$

with terminal condition,

$$V(w, T; T) = \frac{{W_T}^{1-c} - q}{1-c} \ . \quad (24)$$

Observing the homogeneity of the objective function, the restriction and the terminal condition, we conjecture that the value function V must be linear to $\frac{{W_T}^{1-c}-q}{1-c}$.

Let $\quad V(w, t; T) = g(t; T)\frac{w^{1-c}-q}{1-c}$,

and with;

$$V_t = \frac{w^{1-c}-q}{1-c}g'; \quad V_W = w^{-c}g; \quad V_{WW} = -cW^{-1-c}g; \quad (25)$$

equation (23) becomes;

$$\frac{w^{1-c}-q}{1-c}g' + \left\{\begin{matrix} \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha\right]w^{-c}g \\ +({\pi_t}^2\sigma^2 2\sigma\rho\beta\pi_t)(-cW^{-1-c}g) \end{matrix}\right\} = 0, \quad (26)$$

$$\frac{w^{1-c}-q}{1-c}g' + \left\{\begin{matrix} \left[w(\varepsilon + \sigma t) + \pi_t\big(\mu - (\varepsilon + \sigma t)\big) + \alpha\right]w^{-c}g \\ -c({\pi_t}^2\sigma^2 2\sigma\rho\beta\pi_t)w^{-1-c}g \end{matrix}\right\} = 0, \quad (27)$$

the new H.J.B equation becomes.

To obtain the optimal value $\pi_t^*$of π_t, we differentiate (27), with respect to π_t and evaluate to obtain:$[\mu - (\varepsilon + \sigma t)]w^{-c}g - c[\pi_t\sigma^2 + \rho\beta]w^{-1-c}g = 0$

$$c\pi_t\sigma^2 w^{-1-c}g = \big(\mu - (\varepsilon + \sigma t)\big)w^{-c}g - c\sigma\beta\rho w^{-1-c}$$

$$\pi_t^* = \frac{[\mu-(\varepsilon+\sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}. \quad (28)$$

This is the insurance company's optimal investments in risky asset, stock, that is both horizon and wealth dependent.

Lemma 1: The HJB equation (27) has the solution given as;

$$g = e^{K\int_t^T \emptyset(\tau)d\tau} \quad (29)$$

Proof:

The solution of the HJB (27) is thus; replacing S_t by its optimal value and rearranging, we obtain;

$$\frac{W^{1-c}-q}{1-c}g' + \left\{(\xi + \sigma t)w + \left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)[(\varepsilon + \sigma t) + \alpha]\left[\frac{[\mu - (\alpha + \sigma t)]W}{\gamma\sigma^2}\right]w^{-c}\right\}g$$
$$- \quad \frac{cw^{-1-c}}{2}\left\{\left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)^2\sigma^2 + \beta^2 + 2\left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)\beta\sigma\rho + \right\}g = 0,$$

$$\frac{w^{1-c}-q}{1-c}g' + \left\{\left\{(\xi + \sigma t)w + \left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)[(\varepsilon + \sigma t) + \alpha]\left[\frac{[\mu - (\alpha + \sigma t)]W}{\gamma\sigma^2}\right]w^{-c}\right\}\right.$$
$$\left. - \quad \frac{cw^{-1-c}}{2}\left\{\left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)^2\sigma^2 + \beta^2 + 2\left(\frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)\beta\sigma\rho + \right\}\right\}g = 0$$

$$\frac{w^{1-c}-q}{1-c}g' + \emptyset(t)g = 0, \quad (30)$$

where

$$\emptyset(t) = \left\{\left\{(\xi + \sigma t)w + \left(\frac{[\mu-(\varepsilon+\sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)[(\varepsilon + \sigma t) + \alpha]\left[\frac{[\mu-(\alpha+\sigma t)]W}{\gamma\sigma^2}\right]w^{-c}\right\} - \quad \frac{cw^{-1-c}}{2}\left\{\left(\frac{[\mu-(\varepsilon+\sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)^2\sigma^2 + \beta^2 + 2\left(\frac{[\mu-(\varepsilon+\sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}\right)\beta\sigma\rho + \right\}\right\}.$$

We obtain the integral of the differential equation of the function g as;

$$\int \frac{g'}{g} = k\int_t^T \emptyset(\tau)d\tau,$$

where$k = \frac{c-1}{w^{1-c}-q}$ and $w^{1-c} \neq q$.

That is;$\ln g + s = k\int_t^T \emptyset(\tau)d\tau$

$$g(t; T) = e^{K\int_t^T \emptyset(\tau)d\tau}, \quad (31)$$

to which we obtain with the terminal condition ,

$$g(T, T) = e^{K\int_T^T \emptyset(\tau)d\tau} = 1. \quad (32)$$

This implies that the horizon dependent solution to the insurance company's investment problem is:

$$V(w,t,T) = \frac{w^{1-c}-q}{1-c} e^{K\int_t^T \emptyset(t)d\tau}, c \neq 1 . \quad (33)$$

This is the maximized lifetime expected utility at time t under optimal investment policy, and at terminal date,T,$V(w,T,T) = \frac{w^{1-c}-q}{1-c}$ as expected.

4. Findings

The findings of this research are as follows;

- The optimized investment in the risky asset (stock) is horizon dependent and the wealth.

$$\pi_t^* = \frac{[\mu - (\varepsilon + \sigma t)]w}{c\sigma^2} - \frac{\beta\rho}{\sigma}$$

- The optimized value of the insurance company is also horizon dependent;

$$V(w,t,T) = \frac{w^{1-c}-q}{1-c} e^{K\int_t^T \emptyset(t)d\tau}, c \neq 1.$$

5. Conclusion

In this study, the problem of optimizing investment returns of an insurance company with time–varying rate of return was dealt with for power and utility function. The basis for this work is on the fact that risk reserve process of an insurance company can weakly converge to a Brownian motion process. The main emphasis has been on how the utility function affects the insurance company's portfolio selection given investment choices. The proportions for optimizing the company's expected return was observed to be constant proportions of the investor's total wealth.

6. Recommendation

The findings of this work show that the value function and the investments are horizon dependent, so the managers of the assets of the insurance company should take this into consideration horizon dependency when making policy decisions.

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