# Five Dimensional Space-Time in General Relativity 

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## Research Article

$\boldsymbol{A} \boldsymbol{b s t r a c t}$ : In this paper we have deduced the plane wave solutions of field equations $R_{i j}=0$ in five dimensional space - time $V_{5}$ in general relativity on the line of Takeno (1961).
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## 1 Introduction

In general relativity, the plane gravitational waves namely $(z-t)$ - type and $(t / z)$ - type in $V_{4}$ are mathematically exposed by Takeno (1961). In this paper we have extended his investigations to five dimensional space $V_{5}$. There is nothing unphysical in supposing higher dimensional space - time (as a matter of fact the recent string theories could only be talked about if the space time is 10 - dimensional or 20 -dimensional). In this paper we have studied the $(z-t)$ - type and $(t / z)$ - type waves in $V_{5}$ and noted that the expressions for various quantities obtained by Takeno (1961) are retained in their forms here also. From our investigations, all the results of Takeno (1961) can be obtained by assigning appropriate values to the functions concerned.

## 2 Plane Wave Solutions

We consider a five dimensional space - time $V_{5}$ and formulated Takeno's
definition of a plane wave as follows:
A plane wave $g_{i j}$ is a non-flat solution of the field equations

$$
\begin{equation*}
R_{i j}=0 \quad(\mathrm{i}, \mathrm{j}=1,2,3,4,5) . \tag{2.1}
\end{equation*}
$$

In an empty region of the space - time such that

$$
\begin{equation*}
g_{i j}=g_{i j}(Z), Z=Z\left(x^{i}\right), \text { where } x^{i}=x, y, u, z, t \tag{2.2}
\end{equation*}
$$

in some suitable coordinate system and

$$
\begin{align*}
& g^{i j} Z,,_{i},_{j}=0 \text { where } Z,_{i}=\frac{\partial Z}{\partial x^{i}}, \text { etc. }  \tag{2.3}\\
& Z=Z(z, t), \quad\left(Z,_{4} \neq 0 ; Z,_{5} \neq 0\right) \tag{2.4}
\end{align*}
$$

In this definition the signature convention adopted as follows

$$
g_{a a}<0,\left|\begin{array}{ll}
g_{a a} & g_{a b} \\
g_{b a} & g_{b b}
\end{array}\right|>0,\left|\begin{array}{lll}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{array}\right|<0,
$$

$\left|\begin{array}{llll}g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44}\end{array}\right|>0, g_{55}>0$
( not summed for a and $\mathrm{b} ; \mathrm{a}, \mathrm{b}=1,2,3,4$ ) and accordingly

$$
\begin{equation*}
g=\operatorname{det}\left(g_{i j}\right)>0 \tag{2.6}
\end{equation*}
$$

Though the definition of a plane wave defined by us is similar to that of H. Takeno, the result (2.5) is opposite to the corresponding result of Takeno which is obvious because of the signature difference. From equations (2.3) and (2.4) we have

$$
\begin{equation*}
g^{44} \varphi^{2}+2 g^{45} \varphi+g^{55}=0, \tag{2.7}
\end{equation*}
$$

where $\varphi=Z,{ }_{4} / Z,{ }_{5}$,which on simplifying and solving

$$
\text { gives } t+z \phi=\omega
$$

$\omega$ being an arbitrary function of $Z$.
In the notations of Takeno, we obtain
$Z,_{4}=\frac{\varphi}{M} ; \quad Z_{5}=\frac{1}{M} ; M_{,_{4}}=\frac{\varphi N}{M}-\bar{\varphi} ; M,_{5}=\frac{N}{M}$,
where $M=\bar{\omega}-z \bar{\omega} \neq 0 ; N=\overline{\bar{\omega}}-z \overline{\bar{\varphi}}$.
And $\operatorname{bar}(-)$ over a letter denotes derivative with respect to Z .

The Christofels symbols of second kind assume the values as follows:

$$
\begin{aligned}
& 2 M \Gamma_{a b}^{i}=-g_{a b} \omega^{i}, \text { for } a=b \\
& 2 M \Gamma_{a 4}^{i}=\varphi g^{i j} \bar{g}_{a j} \text { for } i=j=a \\
& 2 M \Gamma_{a 5}^{i}=g^{i j} \bar{g}_{a j} \text { for } i=j=a \\
& 2 M \Gamma_{44}^{i}=2 \phi g^{i j} \bar{g}_{4 j}-\bar{g}_{44} \omega_{\text {for }}^{i} i=j=4,5
\end{aligned}
$$

$2 M \Gamma_{55}^{i}=2 g^{i j} \bar{g}_{5 j}-\bar{g}_{55} \omega_{\text {for }}^{i} i=j=4,5$
$2 M \Gamma_{45}^{i}=g^{i j}\left(\bar{g}_{4 j}+\phi \bar{g}_{5 j}\right)-\bar{g}_{45} \omega_{\text {for }}^{i} i=j=4,5$,
where $\mathrm{a}, \mathrm{b}=1,2,3$ and $\omega^{i}=\phi g^{4 i}+g^{5 i}$.
Noting $\omega^{\mathrm{i}}$, equation (2.7) reduces to
$\left(g^{44} \phi+g^{45}\right) \phi+\left(g^{45} \phi+g^{55}\right)=0$
$\omega^{4} \phi+\omega^{5}=o$
The field equation $R_{i j}=0$ then yields

$$
\begin{align*}
& \rho_{a}=\bar{g}_{a i} \omega^{i}=0 \quad(a=1,2,3)  \tag{2.9}\\
& R_{a \alpha}=0,(\alpha=4,5)
\end{align*}
$$

Where,
$R_{a b}=\frac{\rho_{a} \rho_{b}}{2 M^{2}} \quad, \quad R_{\alpha \beta}=\frac{N \rho_{a \beta}}{M^{3}}+\frac{\sigma_{a \beta}}{M^{2}}$
and the equation s $R_{i j}=0$ becomes
$N \rho_{\alpha \beta}+M \sigma_{\alpha \beta}=0$
( $a, b=1,2,3$, and $\alpha, \beta=4,5$ )
Substituting the values of N and M , above equations become

$$
\begin{equation*}
\overline{\bar{\omega}} \rho_{\alpha \beta}+\bar{\omega} \sigma_{\alpha \beta}=0=\overline{\bar{\phi}} \rho_{\alpha \beta}+\bar{\phi} \sigma_{\alpha \beta} \tag{2.12}
\end{equation*}
$$

where

$$
\begin{aligned}
& \sigma_{44}=-\bar{\rho}_{44}+\frac{\left(\phi^{2} L_{1}-4 \phi L_{2} \rho_{4}+2 \rho_{4}^{2}\right)}{4} \\
& \sigma_{55}=-\bar{\rho}_{55}+\frac{\left(L_{1}-4 L_{2} \rho_{5}+2 \rho_{5}^{2}\right)}{4} \\
& \rho_{44}=-\phi^{2} L_{2}+\phi \rho_{4}, \rho_{55}=-L_{2}+\rho_{5} \\
& \rho_{45}=\rho_{54}=-\phi L_{2}+\frac{\rho_{4}}{2}+\frac{\phi \rho_{5}}{2}
\end{aligned}
$$

with

$$
\rho_{i}=\bar{g}_{i j} \omega^{i}, \mathrm{~L}_{2}=\overline{\log \sqrt{-g}}, L_{1}=g^{i j} g^{k l} \bar{g}_{i k} \bar{g}_{j l} .
$$

It is noted that Takeno's forms of the various quantities are retained in $\mathrm{v}_{5}$ too. Thus the plane wave solutions exists in higher dimensional space - time $\mathrm{v}_{5}$ and are given by $\mathrm{g}_{\mathrm{ij}}$ satisfying equations ( 2.2 ), ( 2.5 ) (2.9) and (2.11) .

1. $(z-t)-$ type $)$ and $(t / z)-$ type waves

If we consider $\mathrm{Z}=\mathrm{z}-\mathrm{t}$, we get
$\phi=Z,{ }_{4} / Z,{ }_{5}=1 /-1=-1$,
$\omega=t+z \phi=-Z$,
$M=-1$ and $N^{\prime}=0$.

Thus for $\mathrm{Z}=(z-t)$, the field equations (2.11) reduce to
$\sigma_{\alpha \beta}=0$
Similarly considering $Z=t / z$, we obtain
$\phi=\frac{-t / z^{2}}{1 / z}=-Z$,
$\omega=t+\frac{t}{Z}(-Z)=0$
$M=0-z(-1)=z$ and $N=0-z(0)=0$.
Hence, for $Z=t / z$, the field equations (2.11) become $\sigma_{\alpha \beta}=0{ }_{\text {since }} z \neq 0$
Furthermore, $\sigma_{\alpha \beta}=0$ reduces to
$\bar{L}_{2}-\bar{\rho}_{5}+\frac{\rho_{5}^{2}}{2}-L_{2} \rho_{5}+\frac{L_{1}}{4}=0$,
for both $(z-t)$ and $(t / z)-$ type waves.
Choosing a coordinate system
$g_{i j}=0 \quad$ then $g^{i j}=0, \quad(i \neq j)$.
we have
$\omega^{i}=\phi g^{4 i}+g^{5 i}=\left(0,0,0, \phi g^{44}, g^{55}\right)$
and equation (2.8) reduce to
$\phi^{2} g_{55}+g_{44}=0$.
Noting $g_{i j}=0$ and equation (3.2) we write
$\left[g_{i j}\right]=\left[\begin{array}{ccccc}-A & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 \\ 0 & 0 & -A & 0 & 0 \\ 0 & 0 & 0 & -C & 0 \\ 0 & 0 & 0 & 0 & C\end{array}\right]$
$\left[g^{u}\right]=\left[\begin{array}{ccccc}\frac{-1}{A} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{A} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{A} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{C} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{C}\end{array}\right]$
Where $A>0, C>0$ and $g=\operatorname{det}\left(g_{i j}\right)=m c^{2}$,
Where $m=-A^{3}$
For $(z-t)$ - type wave the line element assumes the form
$d s^{2}=-A\left(d x^{2}+d y^{2}+d u^{2}\right)-C\left(d z^{2}-d t^{2}\right)$
where $A, C$ are functions of $Z=(z-t)$ and we obtain
$\omega^{i}=\left(0,0,0, \frac{1}{C}, \frac{1}{C}\right)$
$\rho_{i}=\left(0,0,0, \frac{-\bar{C}}{C}, \frac{-\bar{C}}{C}\right)$
$L_{2}=\frac{\bar{m}}{2 m}+\frac{\bar{C}}{C}$
$L_{1}=\frac{\bar{m}^{2}}{m^{2}}+2 \frac{\bar{C}^{2}}{C^{2}}-2 \frac{\sum}{m}$
with $\sum=-A \bar{A}^{2}-A \bar{A}^{2}-A \bar{A}^{2}=-3 A \bar{A}^{2}$.
The field equation (3.1) reduces to
$\left(\frac{\bar{m}}{2 m}+\frac{\bar{C}}{C}\right)-\left(\frac{\overline{\bar{C}}}{C}\right)+\frac{\bar{C}^{2}}{2 C^{2}}-\left(\frac{\bar{m}}{2 m}+\frac{\bar{C}}{C}\right) \frac{\bar{C}}{C}+\frac{\bar{m}^{2}}{4 m^{2}}+\frac{\bar{C}^{2}}{2 C^{2}}-\frac{\sum}{2 m}=0$
i.e. $\frac{1}{2}\left[\frac{m \overline{\bar{m}}-\bar{m}^{2}}{m^{2}}\right]+\frac{C \overline{\bar{C}}-\bar{C}^{2}}{C^{2}}-\frac{C \overline{\bar{C}}-\bar{C}^{2}}{C^{2}}+\frac{\bar{C}^{2}}{2 C^{2}}-\frac{\overline{m \bar{C}}}{2 m C}-\frac{\bar{C}^{2}}{C^{2}}+\frac{\bar{m}^{2}}{4 m^{2}}+\frac{\bar{C}^{2}}{2 C^{2}}-\frac{\Sigma}{2 m}=0$
i.e. $\frac{\bar{m}}{2 m}-\frac{\bar{m} \bar{C}}{2 m C}-\frac{\bar{m}^{2}}{4 m^{2}}-\frac{\Sigma}{2 m}=0$,
which is again in Takeno's form.

For $(t / z)-t y p e$ wave, the line element reduces to
$\left.d s^{2}=-A\left(d x^{2}+d y^{2}+d u^{2}\right)-C Z^{2} d z^{2}-C d t^{2}\right)$
where $A, C$ are functions of $Z=(t / z)$.
Furthermore we get
$\omega^{i}=\left(0,0,0, \frac{1}{Z C}, \frac{1}{C}\right)$
$\rho_{i}=\left(0,0,0, \frac{-2 C-Z \bar{C}}{C}, \frac{\bar{C}}{C}\right)$,
$L_{2}=\frac{\bar{m}}{2 m}+\frac{\bar{C}}{C}+\frac{1}{Z}$,
$L_{1}=\frac{\bar{m}^{2}}{m^{2}}+2 \frac{\bar{C}^{2}}{C^{2}}-2 \frac{\sum}{m}+\frac{4 \bar{C}}{Z C}+\frac{4}{Z^{2}}$
The field equation (3.1) takes the form
$\frac{1}{2}\left(\frac{m \overline{\bar{m}}-\bar{m}^{2}}{m^{2}}\right)+\frac{C \overline{\bar{C}}-\bar{C}^{2}}{C^{2}}-\frac{1}{Z^{2}}-\frac{C \overline{\bar{C}}-\bar{C}^{2}}{C^{2}}+\frac{\bar{C}^{2}}{2 C^{2}}-\left(\frac{\overline{m \bar{C}}}{2 m C}+\frac{\bar{C}^{2}}{C^{2}}+\frac{\bar{C}}{Z C}\right)=0$
i.e. $\frac{\bar{m}}{2 m}-\frac{\bar{m} \bar{C}}{2 m C}-\frac{\bar{m}^{2}}{4 m^{2}}-\frac{\sum}{2 m}=0$
which is again in Takeno's form.

## 4. The cosmological constant and plane gravitational waves

It has been established that the cosmological constant $\wedge$ contributes nil to the plane gravitational waves in $\mathrm{v}_{4}$. This also holds for higher dimensional space - time $v_{5}$ and this fact have been established as follows.
Noting equations (2.10), the field equation for
$a, b=1,2,3$, give
$\rho_{a} \rho_{b}=2 \wedge M^{2} g_{a b}$
giving
$\left(\rho_{a}\right)^{2}=2 \wedge M^{2} g_{a a}$
and
$\left(\rho_{b}\right)^{2}=2 \wedge M^{2} g_{b b}$ (not summed for a and $\mathrm{b}: \mathrm{a}, \mathrm{b}=1,2,3$ )
Which further yield

$$
\left(\rho_{a}\right)^{2}\left(\rho_{b}\right)^{2}=4 \wedge^{2} M^{4}\left(g_{a b}\right)^{2} \text { for } a=b
$$

Then one can easily deduce that

$$
\left(2 \wedge M^{2} g_{a a}\right)\left(2 \wedge M^{2} g_{b b}\right)=4 \wedge^{2} M^{4}\left(g_{a b}\right)^{2} \text { for }
$$

$a=b$
or
$\left|\begin{array}{ll}g_{a a} & g_{a b} \\ g_{b a} & g_{b b}\end{array}\right|=0$ (not summed for a and b),
which violates equation (2.5). Thus from the study of the plane gravitational waves in the empty space - time, field equation ultimately reduce to equations $\mathrm{R}_{\mathrm{ij}}=\mathrm{o}$ i.e. $\Lambda=0$. Hence it seems that the introduction of the cosmological terms $\Lambda$ in the field equations of general relativity is not necessary as far as its role in the study of plane gravitational waves in $v_{5}$ is concerned.

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