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Finite Difference Method for Laplace Equation

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Research Article

Abstract: In this paper solution of Laplace equation with Dirichlet boundary and Neumann boundary is discussed by Finite difference method.

Keywords: Laplace Equation, Dirichlet boundary, Neumann boundary, Finite Difference Approximation.

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1. Introduction

Laplace equation is a second order partial differential equation (PDE) that appears in many areas of science and engineering, such as electricity, fluid flow, and steady heat conduction. Solution of this equation, in a domain, requires the specification of certain conditions that the unknown function must satisfy at the boundary of the domain. When the function itself is specified on a part of the boundary, we call that part the Dirichlet boundary; when the normal derivative of the function is specified on a part of the boundary, we call that part the Neumann boundary. In a problem, the entire boundary can be Dirichlet or a part of the boundary can be Dirichlet and the rest Neumann. Consider Laplace equation in a domain D as

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

With boundary condition u =f on S_D and $\frac{\partial u}{\partial n} = g$ on S_N, where *n* is the normal to the boundary, S_D is the Dirichlet boundary, and S_N is the Neumann boundary.

A typical problem is schematically shown in Fig1.1

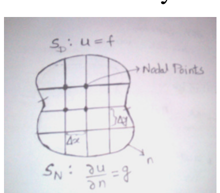

Figure 1.1

Finite difference methods for partial differential equations are studied in [1],[2],[3],[4],[5],[6]. Idea of finite difference method is to descretize the partial differential equation by replacing partial derivatives with their approximation that is finite differences. In this method, the PDE is converted into a set of linear, simultaneous equations. Which are written in the matrix equation and then solution is obtained by solving the matrix equation or solution can be obtained by solving simultaneous equations iteratively. In this paper we use the iterative technique.

2. Finite Difference Method

Let us divide a two dimensional region into the points with increments in the x and y direction as Δx and Δy where Δx=Δy=h

Each nodal point is designated by a numbering scheme i and j where i and j are x and y increments respectively as shown in fig.2.1. We have centre difference approximation for second order derivatives as

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{(i,j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2) \tag{2.1}$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{(i,j)} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} + o(h^2) \tag{2.2}$$

By combining (2.1) and (2.2), we have

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{(i,j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0$$

Therefore

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) \tag{2.3}$$

This shows that value of u at any point of the domain D is the average of the surrounding points in the five point stencil of the following fig.2.1

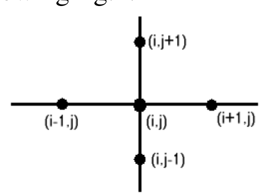

Figure 2.1

3. Dirichlet Problem

Consider the Fig.3.1 containing only four interior points, u is given on the East, West, North and South walls.

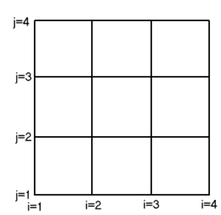

Figure 3.1

Thus the boundary points u(1,2), u(1,3),u(2,4), u(3,4), u(4,3), u(4,2), u(3,1) and u(2,1) are known. We have to calculate the interior points u(2,2), u(3,2), u(2,3), u(3,3).
We begin the iterative process by assuming
$u^{(0)}(2,2)= u^{(0)}(3,2)= u^{(0)}(2,3)= u^{(0)}(3,3)=0$
The superscript (0) denote the 0th iteration.
Using Gauss-Seidel iteration we obtain the 1st iteration for interior points starting at bottom left as

$$\begin{aligned}
u^{(1)}(2,2) &= \tfrac{1}{4}\left(u(1,2)+u^{(0)}(3,2)+u(2,1)+u^{(0)}(2,3)\right)\\
u^{(1)}(3,2) &= \tfrac{1}{4}\left(u^{(1)}(2,2)+u(4,2)+u(3,1)+u^{(0)}(3,3)\right)\\
u^{(1)}(2,3) &= \tfrac{1}{4}\left(u(1,3)+u^{(0)}(3,3)+u^{(1)}(2,2)+u(2,4)\right)\\
u^{(1)}(3,3) &= \tfrac{1}{4}\left(u^{(1)}(2,3)+u(4,3)++u^{(1)}(3,2)+u(3,4)\right)
\end{aligned} \tag{3.1}$$

We compute the first iterated values $u^{(1)}(2,2)$. And use it in the calculation of the first iteration of $u^{(1)}(3,2)$, and so on.
The algorithm for the iterative solution at the interior points on N x M grid

For i=2,N-1
For j=2, M-1
$$u_{i,j} = \tfrac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right) \tag{3.2}$$
End Do
End Do

The corner values u(1,1),u(N,1),u(1,M),u(N,M) are computed from

$$\begin{aligned}
u(1,1) &= \frac{1}{2}[u(1,2)+u(2,1)]\\
u(N,1) &= \tfrac{1}{2}[u(N-1,1)+u(N,2)]\\
u(1,M) &= \frac{1}{2}[u(1,M-1)+u(2,M)]\\
u(N,M) &= \frac{1}{2}[u(N,M-1)+u(N-1,M)]
\end{aligned} \tag{3.3}$$

Thus

- The boundary values (except corner values) are known from boundary conditions
- Corner values are obtained from equation (3.3)
- All other values are iteratively computed from(3.1) or (3.2)

4. Neumann Problem

The Neumann problem is posed on the grid of fig.4.1

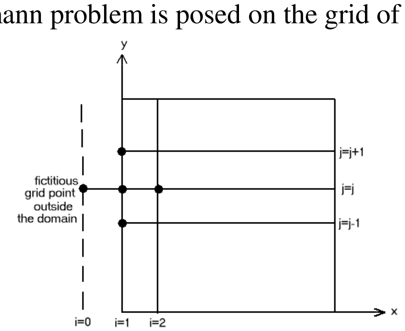

Figure 4.1

The Dirichlet condition is specified on the north, east and south walls. On the west wall Neumann condition is specified as:

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} = g(y)$$

We have

$$\left(\frac{\partial u}{\partial x}\right)_{(i,j)} = \frac{u_{(i+1,j)}-u_{(i-1,j)}}{2h} + o(h^2)$$

Therefore

$$\left(\frac{\partial u}{\partial x}\right)_{(1,j)} = \frac{u_{(2,j)}-u_{(0,j)}}{2h} = -g(1,j) \tag{4.1}$$

where (0,j) is the fictitious grid point out side the domain of the problem.
Laplace equation (2.3)at the point (1,j) becomes

$$u_{1,j} = \frac{1}{4}\left(u_{2,j} + u_{0,j} + u_{1,j+1} + u_{1,j-1}\right) \tag{4.2}$$

From (4.1) , we have
$$u(0,j)=u(2,j)+2hg(1,j) \tag{4.3}$$
Substituting value of u(0.j) from(4.3) in(4.2) we get

$$u(1,j) = \tfrac{1}{4}[2u(2,j) + 2hg(1,j) + u(1,j+1) + u(1,j-1)] \tag{4.4}$$

We use equation (4.4) for 2≤j≤M-1 where g(1,j) is a specified function.
Thus boundary values except corner points & west boundary are known from Dirichlet boundary condition specified on north, east and south.
The algorithm for the iterative solution is:
For i=1, N-1
For j=2,M-1
If i=1 then

$$u(1,j) = \tfrac{1}{4}[2u(2,j) + 2hg(1,j) + u(1,j+1) + u(1,j-1)] \tag{4.5}$$

Else

$$u(i,j) = \tfrac{1}{4}[2u(i+1,j) + 2hg(i,j) + u(i,j+1) + u(i,j-1)]$$

End Do
End Do
The values of u at the corner points are computed by using eqn.(3.3)
Thus

- The values u[(i,M), 2 ≤ i ≤ N-1], u[N,j), 2 ≤ j ≤ M-1], u[(i,1), 2 ≤ i ≤ N-1] are known from boundary condition.
- Corner values are obtained from eqn (3.3).
- All other values are itratively computed from eqn.(4.5)

Example 1

A Dirichlet problem is posed on a square domain as follows:

$u_{North} = 100$, $u_{East} = 50$, $u_{South} = 0$, $u_{West} = 75$

When the problem is solved on a 4 x 4 grid, by using Gauss-Seidel Iterative scheme, eight iterations are necessary to reach a solution. Progress of the iterative scheme is shown in Table 1.

Table 1:

87.5	100	100	75.00
75.00	48.43 66.02 70.41 71.50 71.84 71.62 71.42 71.76	53.90 62.69 64.89 65.43 65.60 65.22 65.39 65.56	50.00
75.00	18.75 35.15 43.94 46.14 46.68 46.84 46.47 46.64	17.19 34.77 39.16 40.25 40.53 40.61 40.17 40.12	50.00
37.50	0.00	0.00	25.00

Example 2

A Neumann Problem is posed on square domain as follows

$u_{North} = 100$, $u_{East} = 50$, $u_{South} = 0$, and on u_{west} as

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x} = 20$$

When the problem is solved on a 4 x 4 grid, by using Gauss-Seidel Iterative scheme. The first iteration values are shown in Table2:

Table 2:

60	100.0	100.0	75
30	42.65	40.62	50
20	18.78	12.5	50
10	0.00	0.00	25

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